Remarks on the internal exponential stabilization to a nonstationary solution for 1D Burgers equations
REMARKS ON THE INTERNAL EXPONENTIAL STABILIZATION TO A NONSTATIONARY SOLUTION FOR 1D BURGERS EQUATIONS

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Abstract. The feedback stabilization of the Burgers system to a nonstationary solution using finite-dimensional internal controls is considered. Estimates for the dimension of the controller are derived. In the particular case of no constraint in the support of the control a better estimate is derived and the possibility of getting an analogous estimate for the general case is discussed; some numerical examples are presented illustrating the stabilizing effect of the feedback control, and suggesting that the existence of an estimate in the general case analogous to that in the particular one is plausible.

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1. Introduction

Let $L > 0$ be a positive real number. We consider the controlled Burgers equations in the interval $\Omega = (0, L) \subset \mathbb{R}$:
\begin{equation}
\partial_t u + u \partial_x u - \nu \Delta u + h + \zeta = 0, \quad u|_\Gamma = 0.
\end{equation}
Here $u$ stands for the unknown velocity of the fluid, $\nu > 0$ is the viscosity, $h$ is a fixed function, $\Gamma = \partial \Omega$ stands for the boundary $\{0, L\}$ of $\Omega$, and $\zeta$ is a control taking values in the space of square-integrable functions in $\Omega$, whose support, in $x$, is contained in a given open subset $\omega \subset \Omega$.

Let us be given a positive constant $\lambda > 0$, a continuous Lipschitz function $\chi \in W^{1,\infty}(\Omega, \mathbb{R})$ with nonempty support, and a solution $\hat{u} \in W^{\text{int}}$ of (1) with $\zeta = 0$, in a suitable Banach space $\mathcal{W}^{\text{int}}$. Then, following the procedure presented in [BRS11], we can prove that there exists an integer $M = M(|\hat{u}|_{\mathcal{W}^{\text{int}}})$, a function $\eta = \eta(t, x)$, defined for $t > 0$, $x \in \Omega$, such that the solution $u = u(t, x)$ of problem (1), with $\zeta = \chi P_M \eta$, and supplemented with the initial condition
\begin{equation}
u(u(0, x) = u_0(x)\end{equation}
is defined on $[0, +\infty)$ and satisfies the relation $|u(t) - \hat{u}(t)|_{L^2(\Omega, \mathbb{R})}^2 \leq Ce^{-\lambda t}|u(0) - \hat{u}(0)|_{L^2(\Omega, \mathbb{R})}^2$, provided the norm $|u(0) - \hat{u}(0)|_{L^2(\Omega, \mathbb{R})}$ is small enough. Here $P_M$ is the orthogonal projection in $L^2(\Omega, \mathbb{R})$ onto the subspace $L^2_M(\Omega, \mathbb{R}) := \text{span}\{\sin(i\pi x/L) \mid i \in \mathbb{N}, 1 \leq i \leq M\}$. That is, the internal control $\zeta = \chi P_M \eta$ stabilizes exponentially, with rate $\lambda$, the Burgers system to the reference trajectory $\hat{u}$.

Notice that the support of the control $\zeta$ is necessarily contained in that of $\chi$, and that the control is finite-dimensional. Furthermore, we also know that the control can be taken in feedback form, $\zeta(t) = e^{-\lambda t} \chi P_M \chi Q^{L, \lambda}_n(u(t) - \hat{u}(t))$, for a suitable family of linear continuous operators $Q^{L, \lambda}_n : L^2(\Omega, \mathbb{R}) \to L^2(\Omega, \mathbb{R})$, $t \geq 0$ (cf. [BRS11], section 3.2).

We can see that the dimension $M$ of the range of the controller depends only on the norm $|\hat{u}|_{\mathcal{W}^{\text{int}}}$ of $\hat{u}$ but, up to now no precise estimate is known. In the case $\hat{u}$ is independent of time it is possible to give, for the case of the Navier–Stokes equations, a rather sharp description of its dimension $M$, though the range of the controller depends on $\hat{u}$; see, for example, [RT10, BT11, Bar12, BLT06, BT04] (cf. [BRS11], Remark 3.11(c)). The procedure uses the spectral properties of the Oseen–Stokes system and cannot be (at least not straightforwardly) used in the time-dependent case.

The aim of this paper is to establish some first results concerning the dimension $M$ of the range of the internal stabilizing controller, in the case of a reference time-dependent trajectory $\hat{u}$. Notice that this case is not less important for applications because often we are confronted with external forces $h$ that depend on time.

In the case we impose no restriction on the support of the control, more precisely, if we take $\chi(x) = 1$ for all $x \in \Omega$, then we obtain that is it enough to take
\begin{equation}
M \geq L/\pi(2\nu^{-1}e)^{1/2}(\nu^{-1}|\hat{u}|_{\mathcal{W}^{\text{int}}}^2 + \lambda)^{1/2},
\end{equation}
where $\mathcal{W}^{\text{wk}} \supset \mathcal{W}^{\text{int}}$ is a suitable Banach space. In the case our control is supported in a small subset $\omega = \text{supp}(\chi)$, we can also derive that it is “enough” to take
\begin{equation}
M \geq C_1 e^{C_2(1+(\nu^{-1}\lambda)^{1/2}+(\nu^{-1}\lambda)^{3/2}+\nu^{-1}|\hat{u}|_{\mathcal{W}^{\text{wk}}}+\nu^{-2}|\hat{u}|_{\mathcal{W}^{\text{wk}}^2})}
\end{equation}
where $C_1$ and $C_2$ are constants depending on $\chi$ and $\Omega$. We easily see that the estimate in the case of the support constraint in much less reasonable, if we think about an application. The reason of the gap is that the idea used to derive (3) cannot be (at least not straightforwardly) used under the constraint on the support of the control. So one question arises: can we improve (4)? To derive (4) we depart from an exact
null controllability result, carrying the cost associated with the respective control. For stabilization, with a given (finite) positive rate $\lambda > 0$, we do not need to reach zero; that is why we believe the estimate can be improved, if we can avoid using the exact controllability result.

We have performed some numerical simulations whose results suggest that the possibility of getting, also in the general case, an estimate analogous to (3) is plausible. We focus on the 1D Burgers equations because the simulations are much simpler to perform in this setting. However, we believe that the difficulties to find an estimate for $M$ will be analogous for the 2D and 3D Burgers and Navier–Stokes systems, and for a suitable class of parabolic systems.

The rest of the paper is organized as follows. In section 2 we recall some well-known results and set up our problem. In section 3 we present the first estimates for a lower bound for the suitable dimension $M$ of the controller; section 3.1 deals with the particular case where we impose no restriction on the support of the control and section 3.2 with the general case. In section 4 we present the discretization of our problem and in sections 5 and 6 we present the results of some simulations we have performed. Finally, in section 7 we give a few more comments on the results.

**Notation.** We write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and nonnegative integers, respectively, and we define $\mathbb{R}_r := (r, +\infty)$, for $r \in \mathbb{R}$, and $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$. We denote by $\Omega \subset \mathbb{R}$ a bounded interval. Given a vector function $u : (t, x) \mapsto u(t, x) \in \mathbb{R}$, defined in an open subset of $\mathbb{R} \times \Omega$, its partial time derivative $\partial_t u$ will be denoted by $\partial_t u$. The partial spatial derivative $\partial_n u$ will be denoted by $\partial_n u$.

Given a Banach space $X$ and an open subset $O \subset \mathbb{R}^n$, let us denote by $L^p(O, X)$, with either $p \in [1, +\infty)$ or $p = \infty$, the Bochner space of measurable functions $f : O \to X$, and such that $|f|^p_X$ is integrable over $O$, for $p \in [1, +\infty)$, and such that $\text{ess sup}_{x \in O} |f(x)|_X < +\infty$, for $p = \infty$. In the case $X = \mathbb{R}$ we recover the usual Lebesgue spaces. By $W^{s,p}(O, \mathbb{R})$, for $s \in \mathbb{R}$, we denote the usual Sobolev space of order $s$. In the case $p = 2$, as usual, we denote $H^s(O, \mathbb{R}) := W^{s,2}(O, \mathbb{R})$. Recall that $H^0(O, \mathbb{R}) = L^2(O, \mathbb{R})$. For each $s > 0$, we recall also that $H^{-s}(O, \mathbb{R})$ stands for the dual space of $H^s(O, \mathbb{R}) = \text{closure of } \{ f \in C^\infty(O, \mathbb{R}) | \text{ supp } f \subset O \}$ in $H^s(O, \mathbb{R})$. Notice that $H^{-s}(O, \mathbb{R})$ is a space of distributions.

For a normed space $X$, we denote by $| \cdot |_X$ the corresponding norm, by $X'$ its dual, and by $\langle \cdot, \cdot \rangle_{X',X}$ the duality between $X'$ and $X$. The dual space is endowed with the usual dual norm: $|f|_{X'} = \sup\{ \langle f, x \rangle_{X',X} | x \in X \text{ and } |x|_X = 1 \}$. In the case that $X$ is a Hilbert space we denote the inner product by $\langle \cdot, \cdot \rangle_X$.

Given an open interval $I \subseteq \mathbb{R}$ and two Banach spaces $X, Y$, we write $W(I, X, Y) := \{ f \in L^2(I, X) | \partial_t f \in L^2(I, Y) \}$, where the derivative $\partial_t f$ is taken in the sense of distributions. This space is endowed with the natural norm $|f|_{W(I, X, Y)} := (|f|^2_{L^2(I, X)} + |\partial_t f|^2_{L^2(I, Y)})^{1/2}$. In the case $X = Y$ we write $H^1(I, X) := W(I, X, X)$. Again, if $X$ and $Y$ are endowed with a scalar product, then also $W(I, X, Y)$ is. The space of continuous linear mappings from $X$ into $Y$ will be denoted by $L(X \to Y)$.

If $I \subset \mathbb{R}$ is a closed bounded interval, $C(I, X)$ stands for the space of continuous functions $f : I \to X$ with the norm $|f|_{C(I, X)} = \max_{t \in I} |f(t)|_X$.

$\overline{C[a_1, a_2]}$ denotes a nonnegative function of nonnegative variables $a_j$ that increases in each of its arguments. $C, C_i, i = 1, 2, \ldots$, stand for unessential positive constants.

2. Preliminaries
2.1. Reduction to local null stabilization. We will denote $V := H^1_0(Ω, ℜ)$, $H := L^2(Ω, ℜ)$, $D(Δ) := V \cap H^2(Ω, ℜ)$, and $V' := H^{-1}(Ω, ℜ)$. The space $H$ is supposed to be endowed with the usual $L^2(Ω, ℜ)$-scalar product; the space $V$ with the scalar product $(u, v)_V := (∇u, ∇v)_H$. The space $H$ is taken a pivot space, and $V'$ is the dual of $V$. The inclusions $V ⊂ H ⊂ V'$ are dense, continuous and compact. The space $D(Δ)$ is endowed with the scalar product $(u, v)_{D(Δ)} := (∆u, ∆v)_H$.

Let us denote

$$W^{wk} := L^∞(R_0, L^∞(Ω, ℜ))$$

Fix a function $h$ and suppose that $\hat{u} ∈ W^{wk}$ solves the Burgers system (1), with $ζ = 0$ and initial condition $\hat{u}_0 := \hat{u}(0)$.

Let us be given a Lipschitz continuous function $χ ∈ W^{1,∞}(Ω, ℜ)$ with nonempty support and $λ > 0$. Then, given another function $u_0$ such that $|u_0 − \hat{u}(0)|_H ≤ 0$ is small enough, our goal is to find an integer $M ∈ N_0$ and a control $η ∈ L^2(R_0, H)$ such that the solution of the problem (1)-(2), with $ζ = χE^Ω_0 P^0_M η$ is defined for all $t > 0$ and converges exponentially to $\hat{u}$, that is, for some positive constant $C > 0$ independent of $u_0 − \hat{u}_0$,

$$|u(t) − \hat{u}(t)|^2_H ≤ C e^{-λt}|u_0 − \hat{u}_0|^2_H \text{ for } t ≥ 0.$$  

Here $P^0_M$ stands for the orthogonal projection in $H$ onto the subspace $L^2_M(Ω, ℜ)$ spanned by the first $M$ eigenfunctions $ξ_n$ of the Dirichlet Laplacian in $Ω$, that is,

$$L^2_M(Ω, ℜ) := \text{span}\{ξ_n | n ∈ N_0, n ≤ M\}$$

where $Ω$ is an open interval such that supp($χ$) ⊆ $Ω ⊆ Ω$, and $E^Ω_0 : L^2(Ω, ℜ) → H$ is the extension by zero outside $Ω$, that is $E^Ω_0 f(x) := \left\{ \begin{array}{ll} f(x) & \text{if } x ∈ Ω \\ 0 & \text{if } x ∈ Ω \setminus Ω \end{array} \right.$.

Recall that it is well-known that the complete system of (normalized) Dirichlet eigenfunctions $\{ξ_n | n ∈ N_0\}$ and the corresponding system of vector fields $\{α_n | n ∈ N_0\}$ are given explicitly by

$$ξ_n(x) := √(n”) sin(πx/l), \quad α_n = (π/l)^2 n^2, \quad −Δξ_n = α_n ξ_n, \quad x ∈ Ω,$$

where $l$ stands for the length of $Ω$.

Let us note that, seeking for the control $η$ and considering the corresponding solution $u$, we find that $v = u − \hat{u}$, will solve

$$∂_t v − νΔv + ν∂_x v + ν(∂_x \hat{u}) + ζ = 0, \quad v|_{Γ} = 0, \quad v(0) = v_0,$$

with $ζ = χE^Ω_0 P^0_M η$ and $v_0 = u(0) − \hat{u}(0)$. It is now clear that to achieve (6) it suffices to consider the problem of local exponential stabilization to zero for solutions of (8), where “local” means that the property is to hold “provided $|v_0|_H$ is small enough”.

2.2. Weak and strong solutions. The existence and uniqueness of strong solutions for system (5) can be proved by classical arguments, where weak and strong solutions are understood in the classical sense as in [Lio69, Tem95, Tem01].

Let us introduce, for given Banach spaces $X$ and $Y$, the linear spaces

$$L^2_{loc}(R_0, X) := \{ f | f|_{(0, T)} ∈ L^2((0, T), X) \text{ for all } T > 0 \},$$
$$W_{loc}(R_0, X, Y) := \{ f | f|_{(0, T)} ∈ W((0, T), X, Y) \text{ for all } T > 0 \},$$

and the space

$$W^* := W^{wk} ∩ L^2_{u,loc}(R_0, H^1(Ω, ℜ)),$$
where \( L^2_{u,\text{loc}}(\mathbb{R}_0, H^1(\Omega, \mathbb{R})) \subset L^2_{\text{loc}}(\mathbb{R}_0, H^1(\Omega, \mathbb{R})) \) is the Morrey-like space

\[
L^2_{u,\text{loc}}(\mathbb{R}_0, H^1(\Omega, \mathbb{R})) := \left\{ f \left| \sup_{i \in \mathbb{N}} |f|_{L^2((i,i+1),H^1(\Omega,\mathbb{R}))} < +\infty \right. \right\}.
\]

**Theorem 2.1.** Given \( \hat{u} \in W^{\text{wk}}, \zeta \in L^2((0, T), V'), \) and \( v_0 \in H, \) then there exists a weak solution \( v \in W((0, T), V, V') \) for system (8), in \((0, T) \times \Omega. \) Moreover, \( v \) is unique and depends continuously on the given data \((v_0, \eta):\)

\[
|v|^2_{W((0, T), V, V')} \leq C |v_0|^2_H + |\zeta|^2_{L^2((0, T), H)}.
\]

**Theorem 2.2.** Given \( \hat{u} \in W^{\text{st}}, \zeta \in L^2((0, T), H), \) and \( v_0 \in V, \) then there exists a strong solution \( v \in W((0, T), D(\Delta), H) \) for system (8), in \((0, T) \times \Omega. \) Moreover, \( v \) is unique and depends continuously on the given data \((v_0, \eta):\)

\[
|v|^2_{W((0, T), D(\Delta), H)} \leq C |v_0|^2_V + |\zeta|^2_{L^2((0, T), H)}.
\]

Notice that the proof of the existence and uniqueness of a weak solution can be done following the argument in [Tem01, chapter 3, section 3.2] by using the estimate

\[
|\partial_x (wv)|^2_{L^2} \leq C |w|^2_{L^2(\Omega, R)} |v|^2_{L^2(\Omega, H)} \leq C_1 |w|^2_{H^1(\Omega, R)} |v|^2_{L^2(\Omega, H)} \leq C_2 |w|^2_{L^2} |v|^2_H.
\]

For the existence of a strong solution we can use, in addition, the estimate

\[
|w\partial_x v|^2_{H} \leq |w|^2_{L^2(\Omega, H)} |\partial_x v|^2_{L^2(\Omega, H)} \leq C_3 |w|^2_{L^2} |v|^2_{L^2}.
\]

**Definition 2.1.** We say that \( v \in W_{\text{loc}}(\mathbb{R}_0, V, V') \) is a global weak solution for system (8), in \( \mathbb{R}_0 \times \Omega, \) if \( v|_{(0, T)} \in W((0, T), V, V') \) is a weak solution, for the same system, in \((0, T) \times \Omega, \) for all \( T > 0.\)

**Definition 2.2.** We say that \( v \in W_{\text{loc}}(\mathbb{R}_0, D(\Delta), H) \) is a global strong solution for system (8), in \( \mathbb{R}_0 \times \Omega, \) if \( v|_{(0, T)} \in W((0, T), D(\Delta), H) \) is a strong solution, for the same system, in \((0, T) \times \Omega, \) for all \( T > 0.\)

**Corollary 2.3.** Given \( \hat{u} \in W^{\text{wk}}, \zeta \in L^2_{\text{loc}}(\mathbb{R}_0, V'), \) and \( v_0 \in H, \) then there exists a weak solution \( v \in W_{\text{loc}}(\mathbb{R}_0, V, V') \) for system (8), in \( \mathbb{R}_0 \times \Omega, \) which is unique and there holds estimate (10).

**Corollary 2.4.** Given \( \hat{u} \in W^{\text{st}}, \zeta \in L^2_{\text{loc}}(\mathbb{R}_0, H), \) and \( v_0 \in V, \) then there exists a strong solution \( v \in W_{\text{loc}}(\mathbb{R}_0, D(\Delta), H) \) for system (8), in \( \mathbb{R}_0 \times \Omega, \) which is unique and there holds estimate (11).

Finally notice that system (1)-(2), is a particular case of (8) (with \( \hat{u} = 0 \)), hence Theorems 2.1 and 2.2 and Corollaries 2.3 and 2.4 also hold for (1)-(2) (with \( h + \zeta \) in the role of \( \zeta \)).

### 2.3. Existence of a stabilizing control

We claim that the existence of a suitable integer \( M = C_{[\zeta, \hat{u}, \text{st}]} \) and a suitable control \( \eta \in L^2(\mathbb{R}_0, H), \) such that \( \zeta = \chi_0^\Omega P^0_M \eta \) stabilizes system (3) to zero, can be derived following the procedure presented in [BRST11]. Indeed the procedure in [BRST11] uses two key ingredients: the smoothing property of the Oseen-Stokes system and an observability inequality for the adjoint Oseen-Stokes system.

In our setting we have the Oseen-Burgers system

\[
\partial_t v - \nu \Delta v + \partial_x (\hat{u} v) + \zeta = 0, \quad v|_\Gamma = 0, \quad v(0) = v_0,
\]

and its “time-backward” adjoint

\[
-\partial_t q - \nu \Delta q + \hat{u} \partial_x q + f = 0, \quad q|_\Gamma = 0, \quad q(T) = q_1
\]
for \( q_1 \in H \) and \( f \in L^2((0, T), V') \). The system (12) is a parabolic system and hence has the smoothing property if \( \hat{u} \in W^{1, \infty} \) (cf. [BRS11, Lemma 2.1]); on the other hand the desired internal observability inequalities for (13) can be found, for example in [Ima95, DZZ08, Yam09].

**Remark 2.1.** Theorems 2.1 and 2.2 and Corollaries 2.3 and 2.4 also hold for system (12) in the role of system (8).

**Remark 2.2.** In [BRS11, section A.3] it is considered only the case when we take \( O = \Omega \). However it is straightforward to check that we can repeat the idea taking \( \text{supp}(\chi) \subseteq \bar{O} \subseteq \Omega \). This idea to take a more general subset \( O \) is borrowed from [Rod13]; below in Remark 3.1 we will explain why considering a more general \( O \) may be interesting. Also, in [BRS11] the function \( \chi \) is taken in \( C^1(\Omega, \mathbb{R}) \), and this regularity is used to derive the truncated observability in [BRS11, section A.3]; the same arguments can be followed if \( \chi \) is a continuous function in \( W^{1, \infty}(\Omega, \mathbb{R}) \).

We can then conclude that we have the following results (cf. [BRS11, Theorem 3.1 and Proof of Theorem 3.6]).

**Theorem 2.5.** For given \( \hat{u} \in W^{1, \infty} \), \( v_0 \in H \) and \( \lambda > 0 \), there is an integer \( M = \overline{C}[\lambda, |\hat{u}|_{W^{1,\infty}}] \geq 1 \) and a control \( \eta^{\hat{u}, \lambda}(v_0) \in L^2(\mathbb{R}_0, H) \) such that the solution \( v \) of system (12), with \( \zeta = \chi \mathbb{E}_0^O P^O_M \eta \), satisfies the inequality
\[
|v(t)|^2_H \leq \overline{C}[\lambda, |\hat{u}|_{W^{1,\infty}}] e^{-\lambda t} |v_0|^2_H, \quad t \geq 0,
\]
where \( M \) and the positive constant \( \overline{C}[|\hat{u}|_{W^{1,\infty}}, \lambda] \) in (14) do not depend on \( v_0 \). Moreover, the mapping \( v_0 \mapsto \eta^{\hat{u}, \lambda}(v_0) \) is linear and satisfies the inequality
\[
|\tilde{\eta}^{\hat{u}, \lambda}(v_0)|_{L^2(\mathbb{R}_0,H)}^2 \leq \overline{C}[\lambda, |\hat{u}|_{W^{1,\infty}}, \lambda] |v_0|^2_H, \quad \text{for } 0 \leq \tilde{\lambda} < \lambda.
\]
Furthermore the control can be taken in feedback form
\[
\zeta = e^{-\lambda t} \chi \mathbb{E}_0^O P^O_M ((\lambda Q^{1, \lambda}_u v)|_O)
\]
for a suitable operator \( Q^{1, \lambda}_u : H \to H \), \( t \in \mathbb{R}_0 \).

**Theorem 2.6.** Let \( M = \overline{C}[\lambda, |\hat{u}|_{W^{1,\infty}}] \) be the integer constructed in Theorem 2.5. Then there are positive constants \( \Theta \) and \( \epsilon \) depending only on \( |\hat{u}|_{W^{1,\infty}} \) and \( \lambda \) such that for \( |v_0|_H \leq \epsilon \) the solution \( v \) of system (8), with \( \zeta \) as in (15), is well defined for all \( t \geq 0 \) and satisfies the inequality
\[
|v(t)|^2_H \leq \Theta e^{-M t} |v_0|^2_H \quad \text{for } t \geq 0.
\]

Notice that the feedback rule is found to globally stabilize to zero the linear Oseen–Burgers system (8). Then, Theorem 2.6 says that the same feedback rule also locally stabilizes to zero the bilinear system (8).

### 2.4. Setting of the problem.

The main goal of this work is to provide some first estimates for the integer \( M \) in Theorem 2.5. We already know that \( M = \overline{C}[|\hat{u}|_{W^{1,\infty}}] \) depends on \( \lambda \) and \( |\hat{u}|_{W^{1,\infty}} \). Following [BRS11] the bound for the integer \( M \) in Theorem 2.5 is related to a suitable observability inequality, for the adjoint system (13), of the form
\[
|q(t)|^2_H \leq C|\chi q|^2_{L^2((a, a+T)), L^2(\Omega, \mathbb{R})}
\]
for \( a, T \geq 0 \), where \( C \) is known to depend on \( T \), on \( \Omega \), and on the (support of the) function \( \chi \), then \( M \) will (in principle) also depend on these objects. Furthermore, since
the viscosity coefficient $\nu$ plays a crucial role on the stability of the system, we expect $M$ to depend also on it.

The dependence of $M$ on all these objects is the main focus of this work, in particular the dependence on the triple $(\lambda, |\hat{u}|_{W^{1,\infty}}, \nu)$.

3. ON THE DIMENSION OF THE CONTROLLER

Here we derive some first estimates concerning a lower bound for the integer $M$ in Theorems 2.3 and 2.6.

3.1. The particular case of no support constraint. We consider the case $\mathcal{O} = \Omega$ and $\chi = 1_\Omega$ with $1_\Omega(x) := 1$ for all $x \in \Omega$.

**Theorem 3.1.** If $\chi = 1_\Omega$, in Theorem 2.3 it is sufficient to take

$$M \geq (t/\pi)^{(3e/2\nu)^{1/2}}(\nu^{-1} |\hat{u}|_{W^{1,\infty}}^2 + \lambda)^{1/2},$$

where $e$ is the Napier’s constant.

**Proof.** Let $w$ solve

$$w_t = \nu \Delta w - \partial_x (\hat{u} w) + (\lambda/2) w, \quad w|_\Gamma = 0, \quad w(0) = v_0.$$

By standard arguments we can find

$$\frac{d}{dt}|w|^2_H \leq -2\nu|\nabla w|^2_H + 2|\hat{u}|_{L^\infty(\Omega, \mathbb{R})}|w|_H|\nabla w|_H + \lambda |w|^2_H$$

from which we can derive that

$$|w|^2_{L^\infty((0,T), H)} \leq e^{(t/2\nu)|\hat{u}|_{W^{1,\infty}}^2 + \lambda} |v_0|^2_H.$$  

Now let $\varphi(t) := 1-t/t \in C^1([0,T], \mathbb{R})$, and set $\delta := \varphi w$. Notice that $\delta$ solves

$$\partial_t \delta = \nu \Delta \delta - \partial_x (\hat{u} \delta) + (\lambda/2) \delta + (\partial_t \varphi) w, \quad \delta|_\Gamma = 0, \quad \delta(0) = v_0$$

with $\delta(T) = 0$. Let now $M \in \mathbb{N}_0$ be a positive integer and consider the solution $\delta_M$ for the system

$$\partial_t \delta_M = \nu \Delta \delta_M - \partial_x (\hat{u} \delta_M) + (\lambda/2) \delta_M + (\partial_t \varphi) P_M^\alpha w, \quad \delta_M|_\Gamma = 0, \quad \delta_M(0) = v_0.$$

The difference $d := \delta - \delta_M$ solves

$$\partial_t d = \nu \Delta d - \partial_x (\hat{u} d) + (\lambda/2) d + (\partial_t \varphi)(1 - P_M^\alpha) w, \quad d|_\Gamma = 0, \quad d(0) = 0,$$

from which we can also derive

$$|d|^2_{L^\infty((0,T), H)} \leq e^{(t/2\nu)|\hat{u}|_{W^{1,\infty}}^2 + \lambda} \left( |d(0)|^2_H + (3/4\nu)(|\partial_t \varphi|)(1 - P_M^\alpha)|w|^2_{L^2((0,T), V)} \right)$$

and, from $|w|^2_{L^2((0,T), H)} \leq T |w|^2_{L^\infty((0,T), H)}$ and (19), we can arrive at

$$|d|^2_{L^\infty((0,T), H)} \leq T^{-1} e^{2(\nu^{-1}|\hat{u}|_{W^{1,\infty}}^2 + \lambda)} (3/4\nu)\alpha_M^{-1} |v_0|^2_H.$$
Now, from $\alpha_M = (\lambda \tau/L)^2$ (cf. (7), with $\mathcal{O} = \Omega$), setting $M$ satisfying (17), and recalling that $\delta_M(0) = v_0$ and $\delta_M(T_*) = -d(T_*)$, we find that $\alpha_M \geq (\nu^{-1}\|\hat{u}_M\|_{W^{1,\infty}} + \lambda)/2\nu$, and
\[
|\delta_M(T_*)|^2_H \leq |\delta_M(0)|^2_H.
\]
Furthermore, from (19) and (20) we have that $|\delta_M|_{L^\infty((0,T_*),H)} = |d - d(t)|_{L^\infty((0,T_*),H)} \leq C|\delta_M(0)|^2_H$.

Now, notice that we can consider system (18) in $(T_*, +\infty) \times \Omega$ with $w(T_*) = \delta_M(T_*)$, and repeat the arguments. Recursively, we conclude that in each interval $J_i := (iT_*, (i+1)T_*)$, $i \in \mathbb{N}_0$, we have $|\delta_M((i+1)T_*)|^2_H \leq |\delta_M(iT_*)|^2_H$ and $|\delta_M|_{L^\infty((iT_*, H)} \leq C|\delta_M(iT_*)|^2_H$ (with $C$ independent of $i$). Hence, we can conclude that $|\delta_M|_{L^\infty(R_0, H)} \leq C|v_0|^2_H$.

Next we notice that $v := e^{-(\lambda/\nu)t}\delta_M$ solves (12), in $\mathbb{R}_0 \times \Omega$, with the concatenated control $\zeta = \chi P_M^\Omega(0) e^{-(\lambda/\nu)t}(-T_1^{-1})w = -T_1^{-1} e^{-(\lambda/\nu)t} P_M^\Omega w$, where $w|_{J_1}$ solves (18), in $J_1^x \times \Omega$, with $w(iT_*) = \delta_M(iT_*)$; from (19) and from the boundedness of $\{\delta_M(iT_*) | \delta_M(iT_*) \in L^2(H) \}$, we conclude that the family $\{\delta_M|_{L^\infty(J_1, H)} | i \in \mathbb{N}\}$ is bounded; so we have that $e^{-(\lambda/\nu)t} \zeta \in L^2(\mathbb{R}_0, H)$ for all $\lambda < \lambda$. Finally we observe that $|v(t)|^2_H \leq e^{-\lambda t}|\delta_M|^2_{H}(R_0, H) \leq Ce^{-\lambda t}|v_0|^2_H$.

3.2. The general case. Let $w$ solve the system
\[
\partial_t w = \nu \Delta w - \partial_x (\hat{u}w) + (\lambda/2)w + \chi \hat{\eta}, \quad w|_\Gamma = 0, \quad w(0) = v_0.
\]
To simplify the exposition we rescale time as $t = \tau/\nu$. Then $\hat{w}(\tau) := w(\tau/\nu)$ solves
\[
\partial_t \hat{w} = \Delta \hat{w} - \partial_x (\hat{u}\hat{w}) + (\lambda/2)\hat{w} + \chi \hat{\eta}, \quad \hat{w}|_\Gamma = 0, \quad \hat{w}(0) = 0,
\]
with $\hat{u}, \hat{\lambda}, \hat{\eta} = \nu^{-1}(\hat{u}, \lambda, \eta)$. Next, consider the adjoint system
\[
- \partial_t q = \Delta q + \hat{u}\partial_x q + (\lambda/2)q, \quad q|_\Gamma = 0, \quad q(T) = q_T
\]
with $q_T \in H$ (here with no external force; cf. system (13)). From, for example, DZZ08 Theorem 2.1 and DFCGBZ02 Theorem 2.3 (e.g., reversing time in system (23)), we have that given an open set $\omega \subseteq \Omega$, there exists a constant $C_{\omega, \Omega} > 0$, depending on $\omega$ and $\Omega$, such that for any time $T > 0$, the weak solution $q$ for (23) satisfies
\[
|q(0)|^2_H \leq e^{C_{\omega, \Omega}(1+\frac{1}{T}+T\lambda+\lambda^2\eta+(1+T)|\hat{u}|_{W^{1,\infty}}^2)}|q|_{L^2((-T, T), L^2(\omega, \Omega))}^2.
\]
Proposition 3.2. For every $v_0 \in H$, we can find a control $\hat{\eta} = \hat{\eta}(v_0) \in L^2((-T, T), H)$, driving system (22) to $\hat{w}(T) = 0$ at time $t = T > 0$. Moreover, the mapping $\hat{\eta} : v_0 \mapsto \hat{\eta}(v_0)$ is linear and continuous: $\hat{\eta} \in L^2(\mathcal{L}(H \to L^2((-T, T), H)$, and there is a constant $C_{\chi, \Omega}$ such that
\[
|\hat{\eta}(v_0)|^2_{L^2((-T, T), H)} \leq e^{C_{\chi, \Omega}(1+\frac{1}{T}+T\lambda+\lambda^2\eta+(1+T)|\hat{u}|_{W^{1,\infty}}^2)}|v_0|^2_H.
\]
Sketch of the proof. The proof can be done following the arguments in BRS11. First, from (24) we can derive an observability of the form
\[
|q(0)|^2_H \leq e^{C_{\chi, \Omega}(1+\frac{1}{T}+T\lambda+\lambda^2\eta+(1+T)|\hat{u}|_{W^{1,\infty}}^2)}|\chi q|_{L^2((-T, T), H)}^2
\]
for the solution $q$ of system (23) (cf. [BRS11] eq. (A.8)). Then we can prove the null controllability considering the following minimization problem (cf. [BRS11 Problem 3.3])
\[
J_e(\hat{w}, \hat{\eta}) = |\hat{\eta}|^2_{L^2} + \frac{1}{\epsilon} |\hat{w}(T)|^2_H \to \min; \quad \text{with } (\hat{w}, \hat{\eta}) \text{ solving (22)}.
\]
To prove the linearity we can, next, consider the minimization problem
\[
J_e(\hat{w}, \hat{\eta}) = |\hat{\eta}|^2_{L^2} \to \min; \quad \text{with } (\hat{w}, \hat{\eta}) \text{ solving (22)} \quad \text{and } \hat{w}(T) = 0
\]
(cf. BRS11 Problem 3.4).
Considering the null controllability of linear parabolic equations we also refer to [Ima95, section 2].

Theorem 3.3. In Theorem 2.3 it is sufficient to take

\[ M \geq C_{X,\Omega}^0 e^{C_{X,\Omega}^0 (1+\gamma + \alpha) (1+\gamma + \alpha) \frac{1}{2}} \left| \nabla_{\Omega,\omega_{\text{w}}(\omega_{\text{w}})} v_0 \right|^2 H \]

where \( C_{X,\Omega}^0 = \frac{l}{2} (2 + 2 (L)) \frac{1}{2} |\omega_{\text{w}}(\omega_{\text{w}}) \rangle H \), \( l \) is the length of \( \Omega \), and \( C_{X,\Omega}^0 \) is the constant from (25).

Proof. Let \( \bar{w} \) solve (22) with \( \bar{\eta} = \bar{\eta}(v_0) \), and let \( \bar{w}_M \) be the solution of

\[ \partial_t \bar{w}_M = \Delta \bar{w}_M - \partial_x (\bar{w} \bar{w}_M) + (\lambda/2) \bar{w}_M + H \chi^{\partial}_{0} (\bar{\eta}(v_0) \rangle H) , \quad \bar{w}_M |_{\Gamma} = 0, \quad \bar{w}_M(0) = v_0. \]

Then, the difference \( d := \bar{w} - \bar{w}_M \) solves

\[ \partial_t d = \Delta d - \partial_x (d \bar{w}_M) + (\lambda/2) d + H \chi^{\partial}_{0} (1 - P_M) (\bar{\eta}(v_0) \rangle H) , \quad d |_{\Gamma} = 0, \quad d(0) = 0, \]

and taking the scalar product with \( d \), in \( H \), we can arrive at

\[ \frac{d}{dt} |d|^2 H = -2 |\nabla d|^2 H + 2 |\bar{u}|_{L^{\infty}(\Omega,\omega)} |d|_{H} |\nabla d|_{H} + \lambda |d|_{H}^2 \]

\[ + 2 \chi^{\partial}_{0} (1 - P_M)(\bar{\eta}(v_0) \rangle H), \quad d \rangle H. \]

For the last term we find

\[ \chi^{\partial}_{0} (1 - P_M)(\bar{\eta}(v_0) \rangle H), \quad d \rangle H, V \leq \chi^{\partial}_{0} (1 - P_M)(\bar{\eta}(v_0) \rangle H), \quad d \rangle H \]

and from \( |(1 - P_M)(d)|_{L^2(\Omega,\omega)} \rangle H \leq \alpha^{-1}_M |(1 - P_M)(d)|_{L^2(\Omega,\omega)} \rangle H \leq 2 \alpha^{-1}_M |\chi_{\omega_{\text{w}}(\omega_{\text{w}})} |d|_{H}^2 \]

\[ + |\nabla d|^2_{L^2(\Omega,\omega)} \rangle H \geq (\alpha^4_M + \alpha^4_M) |d|^2_{H} \rangle H, \] where \( \alpha_M = \frac{\pi}{2} \), we find

\[ \chi^{\partial}_{0} (1 - P_M)(\bar{\eta}(v_0) \rangle H), \quad d \rangle H, V \leq \alpha^{-1}_M |\chi_{\omega_{\text{w}}(\omega_{\text{w}})} |d|_{H}^2 \]

\[ + \alpha^{-1}_M D_{X,\Omega} |\bar{\eta}(v_0) \rangle H, \quad d \rangle H. \]

Thus from (28),

\[ \frac{d}{dt} |d|^2 H \geq |\bar{u}|_{L^{\infty}(\Omega,\omega)} |d|^2 H + \lambda |d|_{H}^2 + \alpha^{-1}_M D_{X,\Omega} |\bar{\eta}(v_0) \rangle H, \quad d \rangle H \]

and, using (25), we obtain

\[ |d|^2_{L^{\infty}(0,T),H} \leq \alpha^4_M D^2_{X,\Omega} e^{C\chi_{\alpha}(1+\gamma + \alpha) \frac{1}{2}(\lambda + \|\bar{w}_{\text{w}}\|_{\omega_{\text{w}}(\omega_{\text{w}})}^2)} \left| \nabla_{\Omega,\omega_{\text{w}}(\omega_{\text{w}})} v_0 \right|^2 H \]

with \( C\chi_{\alpha} = \max\{1, \chi_{\alpha}\} \). Now the function \( E(T) = e^{C\chi_{\alpha}(1+\gamma + \alpha) (\lambda + \|\bar{w}_{\text{w}}\|_{\omega_{\text{w}}(\omega_{\text{w}})}^2)} \) takes its minimum when \( T = \tau \), with \( \tau \) defined by \( \frac{1}{2} \tau^2 = (\lambda + \|\bar{w}_{\text{w}}\|_{\omega_{\text{w}}(\omega_{\text{w}})}^2) \). Then, choosing \( T = \tau \), and recalling that \( \bar{w}_M(T) = 0 \), we arrive at

\[ |\bar{w}_M(T)|_{H} \leq \alpha^{-1}_M D^2_{X,\Omega} e^{C\chi_{\alpha}(1+\gamma + \alpha) \frac{1}{2}(\lambda + \|\bar{w}_{\text{w}}\|_{\omega_{\text{w}}(\omega_{\text{w}})}^2)} \left| \nabla_{\Omega,\omega_{\text{w}}(\omega_{\text{w}})} v_0 \right|^2 H \]

Therefore, choosing \( M \in \mathbb{N}_0 \) satisfying (27), and recalling that \( \alpha_M = (\frac{3\pi}{2})^2 \), we have

\[ \alpha_M \geq \frac{D_{X,\Omega}^2 \chi_{\alpha}(1+\gamma + \alpha) \frac{1}{2}(\lambda + \|\bar{w}_{\text{w}}\|_{\omega_{\text{w}}(\omega_{\text{w}})}^2)} \left| \nabla_{\Omega,\omega_{\text{w}}(\omega_{\text{w}})} v_0 \right|^2 H \]

and \( |\bar{w}_M(T)|_{H} \leq |\bar{w}_M(0)|_{H}^2 \). Moreover we can deduce from (25) and (29) that

\[ |\bar{w}_M|_{L^{\infty}(0,T),H} = |\bar{w}_M|_{L^{\infty}(0,T),H} + |d|^2_{L^{\infty}(0,T),H} \leq C |\bar{w}_M(0)|_{H}^2 \]

for a suitable constant \( C \) independent of \( \bar{w}_M(0) \). Recursively, repeating the argument in the time interval \((1/T), \infty)\) with \( \bar{w}(1/T) = \bar{w}_M(1/T) \) in (22), we can conclude that the solution \( \bar{w}_M \) will remain bounded for all time \( T \geq 0 \). That is,

\[ |\bar{w}_M|_{L^{\infty}(0,T),H} \leq |\bar{w}|_{L^{\infty}(0,T),H} \]

Next, we notice that \( v(t) := e^{-\lambda t} \hat{w}_M(\nu t) \) solves \( (12) \), in \( \mathbb{R}_0 \times \Omega \), with the concatenated control \( \zeta = \chi \mathbb{P}_0 P_M^2 (\nu e^{-\lambda t} \hat{v}(v_i)(\nu t)) \mid \partial \), where \( \hat{v}(v_i) \), \( i \in \mathbb{N} \), is the control given in Proposition \( 3.2 \) when we consider system \( (22) \), in \( J_i^* \times \Omega \), with \( J_i^* := (i \nu t_i, (i+1) \nu t_i) \), \( i \in \mathbb{N}_0 \), and \( \hat{w}(i \nu t_i) = \hat{w}_M(i \nu t_i) \), in particular \( \hat{v}(v_i)(\nu t) \) is defined for \( t \in (i \nu t_i, (i+1) \nu t_i) \). We can also conclude from \( (25) \) and from the boundedness of \( \{ |\hat{w}_M(i \nu t_i)| \}_{i \in \mathbb{N}} \) that the family \( \{ |\hat{v}(v_i)| \}_{i \in \mathbb{N}} \) is bounded; so \( e^{(\lambda t_{1/2})/\nu} \) is bounded. Finally we observe that \( |v(t)|^2_H \leq e^{-\lambda t} |\hat{w}_M(\nu t)|^2_{L^\infty(\mathbb{R}_0, H)} \leq C e^{-\lambda t} |v_0|^2_H. \)

**Remark 3.1.** Notice that when we shrink the support of \( \chi \) the constant \( C_{\chi, \Omega} \) in \( (25) \) is expected to increase. This is why taking the length \( l \) of \( \mathcal{O} \) in \( (27) \) can compensate a little the increasing of \( C_{\chi, \Omega} \) in order to get a smaller bound for the number \( M \) of needed controls.

### 3.3. The gap due to the support constraint

Comparing the estimates \( (17) \) and \( (27) \), we see that there is a big gap; the former is proportional to \( (\nu^{-2} |\hat{v}|^2_{W^{1,\infty}} + \nu ^{-1} \lambda)^{1/2} \) and the latter depends exponentially in both \( \nu^{-1} |\hat{v}|_{W^{1,\infty}} \) and \( \nu ^{-1/2} \lambda^{1/2} \). For application purposes the latter is much less convenient, so one question arises naturally: can we improve \( (27) \)? It seems that the idea used to derive \( (17) \) cannot (at least straightforwardly) be applied under the support constraint for the controls. On the other side to derive \( (27) \) we start from an exact null controllability result and carry the cost of the respective control. This means that to improve \( (27) \) we will probably need a different idea.

In section \( 5 \) in order to understand if it is possible to improve \( (27) \), say that we also have an estimate like \( (17) \) under the support constraint, we present results of some numerical simulations comparing the number of controls \( M = M_{\text{need}, \Omega} \), that we need to stabilize the system \( (12) \) to zero, to the reference value

\[
M_{\text{ref}} = (\frac{L}{\pi})^\nu (\nu^{-1} |\hat{v}|^2_{W^{1,\infty}} + \lambda)^{1/2} \in \mathbb{R}_0;
\]

(cf. \( (17) \)). Notice that in the case \( \hat{v} = 0 \), and under no support constraint, we can see that the unstable modes of system \( (18) \) are those defined by the inequality \( \nu \alpha_i < \lambda \), that is, \( i < (\frac{L}{\pi})\nu^{-1/2} \lambda^{1/2} = M_{\text{ref}} \). Thus, in this case, it is enough (and necessary) to take the \( M = \lfloor M_{\text{ref}} \rfloor \) controls in \( \{ \sqrt{3} L \sin(i \pi x/L) | i \in \{1, 2, \ldots, M\} \} \) (taking \( \chi \) and the family of controls considered in section \( 3.1 \)). Here \( \lfloor y \rfloor \in \mathbb{N} \) stands for the biggest integer that is strictly smaller than \( y > 0 \).

### 4. Discretization

To perform the simulations in order to check the stabilization of systems \( (1) \) and \( (12) \), to a reference trajectory \( \hat{u} \) and to 0 respectively, we must discretize those systems with the feedback control \( \zeta \) as in \( (15) \).

#### 4.1. Discretization in space

We use a finite-element based approach. We introduce a regular mesh

\[
\Omega_D := (L/N_x, 2L/N_x, \ldots, (N_x-1)L/N_x)
\]

consisting of the interior points of \( \Omega \) that are multiples of the space step \( h = L/N_x \), with \( 2 \leq N_x \in \mathbb{N} \). As basis functions we take the classical hat-functions \( \phi_i \in V \) defined, for \( x \in \Omega \) and each \( i \in \{1, 2, \ldots, N_x - 1\} \) by \( \phi_i(x) = \begin{cases} 
1 - i + x/h, & \text{if } x \in [(i-1)h, ih]; \\
1 + i - x/h, & \text{if } x \in [ih, (i+1)h]; \\
0, & \text{if } x \notin [(i-1)h, (i+1)h].
\end{cases} \)
Remark 4.1. Notice that $\tilde{u} := \sum_{i=1}^{N_x-1} \overline{u}_i \phi_i$, is a piecewise (affine) linear function that takes the same values as $u$ at the points of the mesh $\Omega_D$. Also notice that, since we are dealing with homogeneous Dirichlet boundary conditions, only the values at interior points are unknown for the solution of our system.

The next step is the weak discretization matrix $L_D$ of a given linear operator $u \in \mathcal{L}(V \rightarrow V')$. We define $L_D$ by the formula

$$\mathcal{v}^{\top}L_D \mathcal{v} = \langle L\tilde{u}, \tilde{v} \rangle_{V',V} \text{ for all } u, \ v \in V.$$  

Of key importance are the identity and the Laplace operator. For the identity operator $Iu = u$, we find that $I_D = [(\phi_i, \phi_j)^H] =: \mathbf{M}$ is the so-called Mass matrix, while for the Laplace operator we find $\Delta_D = -[(\partial_x \phi_i, \partial_x \phi_j)^H] =: -\mathbf{S}$, where $\mathbf{S}$ is the so-called Stiffness matrix. Explicitly we have the tridiagonal matrices

$$\mathbf{M} := \frac{h}{6} \begin{bmatrix} 4 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 4 & 1 & 0 & \ldots & 0 \\ 0 & 1 & 4 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & 4 & 0 \\ 0 & \ldots & 0 & 0 & 1 & 4 \\ \end{bmatrix} \quad \text{and} \quad \mathbf{S} := \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & 0 & \ldots & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 & 2 & -1 \\ 0 & \ldots & 0 & 0 & -1 & 2 \\ \end{bmatrix}.$$  

Next, we recall the reference solution $\tilde{u}$ and discretize the operator $v \mapsto B(\tilde{u})v := \partial_x(\tilde{u}v)$, $v \in V$. We start by noticing that, for an arbitrary $w \in V$, $(\partial_x(\tilde{u}v), w)^H = -\langle \tilde{u}v, \partial_x w \rangle_H$, then we consider the approximation $\tilde{w}v = \sum_{j=1}^{N_x-1} \overline{w}_j \overline{v}_j \phi_j$ of $\tilde{u}v$, and we find $-\langle \tilde{w}v, \partial_x \tilde{w} \rangle_H = \sum_{j=1}^{N_x-1} -\overline{w}_j \overline{v}_j \tilde{w}(\phi_j, \partial_x \phi_i)^H$, and

$$(\partial_x(\tilde{u}v), w)^H \approx \mathcal{w}^{\top}B\mathcal{D}_u\mathcal{v},$$  

with $B$ the bidiagonal matrix

$$B := \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ -1 & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & -1 & 0 & 1 \\ 0 & \ldots & 0 & -1 & 0 \\ \end{bmatrix}.$$
and where, for a given vector $\pi \in \mathcal{M}_{(N_x-1) \times 1}$, $\mathcal{D}_\pi$ denotes the diagonal matrix

$$
\mathcal{D}_\pi := \begin{bmatrix} \pi_1 & 0 & 0 & \ldots & 0 \\
0 & \pi_2 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \pi_{N_x-2} & 0 \\
0 & \ldots & 0 & 0 & \pi_{N_x-1} \end{bmatrix}.
$$

Notice also that, rewriting $v \partial_x v$ as $\frac{1}{2} B \nabla v$, we can discretize $v \partial_x v$ as $\frac{1}{2} B \mathcal{D}_\pi \nabla v$.

To discretize the operators in the feedback control rule in (15), that we can rewrite as

$$
\mathcal{F}_v := B_M^O B_M^{O^*} Q_{\xi u}^{l, \lambda} v
$$

with $B_M^O : H \rightarrow H$, and its adjoint $B_M^{O^*} : H \rightarrow H$ given by

$$
B_M^O \eta := e^{-(\gamma/2)t} \chi \mathbb{E}_0^O P_M^O (\eta|_O), \quad B_M^{O^*} \xi = e^{-(\gamma/2)t} \mathbb{E}_0^O P_M^O ((\xi)|_O),
$$

we start by noticing that what we essentially need is an approximation $\mathcal{F}_v$ of $\mathcal{F}_v$, when we only know the approximation $\overline{\nabla v}$ of $v$.

We will construct $\overline{\nabla v}$ in a few steps. For the multiplication operator $v \mapsto \chi v$ we can of course take $\mathcal{D}_\xi v = \overline{\chi \nabla v}$ as an approximation of $\chi v$. For the orthogonal projection $P_M^O$ we start by noticing that

$$
P_M^O (v|_O) = \sum_{n=1}^M (v|_O, \xi_n)_{L^2(O, \mathbb{R})} \xi_n = \sum_{n=1}^M (v, \mathbb{E}_0^O \xi_n)_{H} \xi_n,
$$

then we can take the approximation $P_M^O (v|_O) \approx \sum_{n=1}^M \left( \mathbb{E}_0^O \xi_n^\top \mathbb{M} \xi_n \right) \xi_n$, from which we set the discrete approximation

$$
P_M \overline{\nabla v} \approx \mathbb{E}_0^O P_M (v|_O) \quad \text{with} \quad P_M := S_M M, \quad \text{and} \quad S_M := \sum_{n=1}^M \mathbb{E}_0^O \xi_n \mathbb{E}_0^O \xi_n^\top.
$$

Finally, the linear operator $Q_{\xi u}^{l, \lambda}$ is, at this moment, unknown and (an approximation) has to be found. Note that denoting by $Q_D = (Q_{\xi u}^{l, \lambda})_D$ the discretization of $Q_{\xi u}^{l, \lambda}$, we may take $M^{-1} Q_D \overline{\nabla v} \approx Q_{\xi u}^{l, \lambda} v$ and discretize the feedback rule (34) as follows: first we take the approximation $B_M^{O^*} \hat{\xi} \approx e^{-(\gamma/2)t} P_M \mathcal{D}_\xi \hat{\xi}$, then from (33) and $(\mathcal{F}_v, w)_H = \left( B_M^{O^*} Q_{\xi u}^{l, \lambda} v, B_M^{O^*} w \right)_H$, we find

$$
(\mathcal{F}_v, w)_H \approx \left( B_M^{O^*} Q_{\xi u}^{l, \lambda} v, B_M^{O^*} w \right)_H = B_M^{O^*} w^\top M B_M^{O^*} Q_{\xi u}^{l, \lambda} v \\
\approx e^{-\lambda} \left( P_M \mathcal{D}_\xi \overline{\nabla v}^\top \mathbb{M} \right) \left( P_M \mathcal{D}_\xi \overline{\nabla v} \right)^{-1} \mathbb{M}^{-1} \left( P_M \mathcal{D}_\xi \overline{\nabla v} \right)^{-1} \mathbb{M}^{-1} \mathbb{Q}_D \overline{\nabla v} \\
= \overline{\mathbb{M}} \mathbb{M}^{-1} e^{-\lambda} \left( P_M \mathcal{D}_\xi \right)^{-1} \mathbb{M} \left( P_M \mathcal{D}_\xi \overline{\nabla v} \right)^{-1} \mathbb{M} \\
= \overline{\mathbb{M}} \mathbb{M}^{-1} \mathbb{R} \mathbb{R}^\top \mathbb{Q}_D \overline{\nabla v}
$$

where

$$
R := \left( \mathbb{M}, P_M \mathcal{D}_\xi \mathbb{M}^{-1} \right)^\top
$$

and $\mathbb{M}$, satisfying $\mathbb{M}^\top \mathbb{M} = \mathbb{M}$, is the Cholesky factor of $\mathbb{M}$ (notice that $\mathbb{M}$ is positive definite). Thus we take $\mathcal{F}_v = \overline{\nabla v}$ with

$$
\mathcal{F} := e^{-\lambda} \mathbb{R} \mathbb{R}^\top \mathbb{Q}_D.
$$
4.2. Discretization in time. For discretization in time of system \(\text{(12)}\), considered in a

time interval \([0, T]\), where \(T\) is a positive real number, we introduce a regular mesh

\[
[0, T)_D := (0, T/N_t, 2T/N_t, \ldots, (N_t - 1)T/N_t, T)
\]

consisting of the points in \([0, T]\) that are proportional to the time step \(k := T/N_t\), with

\(N_t \in \mathbb{N}_0\). Then, any function \(u \in H^1((0, T), V)\) is approximated by the values it takes

in \([0, T)_D \times \Omega_D\), that is, we essentially approximate \(u = u(t, x)\) by a matrix \([u] \in \mathcal{M}(N_t-1 \times (N_t+1)\) whose \(jth\) column is the vector \(u(jk, \cdot)\). Therefore \([u]_{ij} = u(jk, ih)\), for \(i \in \{1, 2, \ldots, N_x - 1\}\) and \(j \in \{0, 1, 2, \ldots, N_t\}\)

4.3. Computation of the discretized feedback rule. Proceeding as in [BRST11 section 3.2], it will follow that the operator \(Q^h_{\hat{u}}\) can be set as a symmetric linear and continuous operator defining the optimal cost \((Q^h_{\hat{u}}v_s, v_s)_H\) for the problem of finding \((v, \eta)\) solving \((\text{(12)})\) with \(\zeta = \chi P_M \eta\) and \(v(s) = v_s\) on \((s, +\infty)\), and minimizing \(|e^{(\lambda_2/2)}\partial_x v|^2_{L^2((s, +\infty), H)} + |e^{(\lambda_2/2)}\eta|^2_{L^2((s, +\infty), H)}\), for any \(s \geq 0\). Moreover, from [BRST11 Remark 3.11(b)], we know that \(Q = Q^h_{\hat{u}}\) solves, for \(t > 0\), the Riccati-like differential equation

\[
\partial_t Q - Q(-\nu \Delta + B(\hat{u})) - (-\nu \Delta + B(\hat{u}))^* Q - QB_M^* B_M^* Q - e^{\lambda t} = 0.
\]

4.3.1. Discretization of the differential Riccati equation. To construct the approximation \(Q_D\), we can look for \(Q_D\) solving

\[
\partial_t Q_D - Q_D X - X^T Q_D - e^{\lambda t} Q_D R R^T Q_D + e^{\lambda t} \nu S = 0, \quad t > 0,
\]

with \(R\) as in \((36)\) and

\[
X = X(t) = M^{-1} \left( \nu S + BD_{\hat{u}(\cdot)} \right).
\]

Equivalently, we can look for \(P = e^{\lambda t} Q_D\) solving

\[
\partial_t P - P X - X^T P - P R R^T P + \nu S + \lambda P = 0, \quad t > 0.
\]

Remark 4.2. Notice that from \(X \bar{v} \approx -\nu \Delta v + B(\hat{u}) v\), we have \((Q(-\nu \Delta + B(\hat{u})) v, w)_H \approx \bar{w}^T X \bar{v}\). Similarly we have \(((\nu \Delta v + B(\hat{u}))^* Q v, w)_H \approx \bar{w}^T X^T Q \bar{v}\), and \((Q F v, w)_H \approx \bar{w}^T Q_D \bar{v}\).

4.3.2. Initialization of the differential Riccati equation. We need to solve system \((41)\) backwards in time, thus the question is: how to initialize the system? Roughly speaking, it seems that we would need to know \(P(+\infty)\), and even if we know this (limit) value it is not clear how we could use it.

Recall that, our main goal is to approach the desired solution \(\hat{u}(t)\) as time \(t\) increases but, in a real application we also want to have an effective controller that, for example, guarantees us that after some time \(t = \hat{T} > 0\) we are indeed closer than we were at initial time \(t = 0\), say, e.g., \(|v(T)|_H^2 \leq 1/2|v(0)|_H^2\). Notice that, from an estimate like \((14)\), we cannot guarantee that after a priori given time \(\hat{T}\) the norm of the solution \(v\) is squeezed. We would need to know more information about the constant \(\bar{C}_{[\chi, [\hat{u}]}_{v, w}^{\text{wk}}\) appearing in that estimate. On the other hand, in applications it is reasonable to think of a problem set for a possibly very long time range \(t \in \mathbb{R}\) but, never for an infinite time range.

Thinking about effectiveness and applications we can suppose we are interested in the evolution for time \(t \in [0, T]\), then we can suppose that for time \(t > T\) our solution is stationary, that is, we may study the same problem but, now we can suppose that \(\hat{u}(t) = \hat{u}(T)\), for all \(t \geq T\).
Now we can find $P_T$ solving the algebraic Riccati equation
\begin{equation}
(P - X(T) + \lambda/2I) + (X(T) + \lambda/2I)^\top P - PRR^\top P + \nu S = 0,
\end{equation}
and we can see that, $P_T$ will solve the autonomous system \((\text{[41]})\) for $t \geq T$ (under the supposition $\dot{u}(t) = \dot{u}(T)$ for $t \geq T$).

Then, it remains to solve \((\text{[41]})\) for time $t \in [0, T]$ with the initial condition $P(T) = P_T$.

4.3.3. Solving the Riccati systems.
• General procedure. To solve the algebraic Riccati system \((\text{[42]})\) we use the software available from \([\text{Ben}]\); in this way we find $P_T$.

To solve (backwards in time) the differential system \((\text{[41]})\), for $t \in [0, T]$, with the initial condition $P(T) = P_T$, we proceed as follows. Recall the mesh $[0, T]_D$ of the interval $[0, T]$, defined in \((\text{[38]})\). We have $P^{N_t} := P(N_t k) = P(T) = P_T$; next, recursively, we construct $P^j := P(jk)$ for $j \in \{0, 1, \ldots, N_t - 1\}$, as follows: we start by rewriting \((\text{[41]})\) as
\begin{equation}
- \partial_t P = P - X + \lambda/2I) + (X + \lambda/2I)^\top P - PRR^\top P + \nu S := R_F(P)
\end{equation}
and, we use the Crank-Nicolson inspired scheme
\begin{equation}
-2/k(P^{j+1} - P^j) = R_F(P^j) + R_F(P^{j+1}),
\end{equation}
from which we obtain $R_F(P^j) - 2/k P^j + R_F(P^{j+1}) + 2/k P^{j+1} = 0$, that is,
\begin{equation}
P^j(-X + \lambda/2I - 1/kI) + (X + \lambda/2I - 1/kI)^\top P^j - P^j R^2 + Z^{j+1} = 0
\end{equation}
with $Z^{j+1} = R_F(P^{j+1}) + 2/k P^{j+1} + \nu S$. Hence $P^j$ solves again an algebraic Riccati equation and we can still use the software in \([\text{Ben}]\).

• Initial guess. The software in \([\text{Ben}]\) (see also \([\text{Ben06}]\)) uses a Newton method to solve an algebraic Riccati equation like \((\text{[42]})\). We have to provide an initial guess $Y_0$ such that $-X(T) + \lambda/2I - RR^\top Y_0^\top Y_0$ is stable. This is of course a nontrivial task and we look for the initial guess in three steps:

(i) We set $M = +\infty$ and $\chi = 1_\Omega$. In this case we can see that $R = (M_c M^{-1})^\top$ and $RR^\top = M^{-1}$. Then from \((\text{[40]})\) we obtain that
\begin{equation}
-X(T) + \lambda/2I - M^{-1}Y_0^\top Y_0 = -M^{-1} (\nu S + B D_{\bar{u}(T)}) + \lambda/2I - M^{-1} Y_0^\top Y_0
\end{equation}
will be stable for $Y_0 = \beta M_c$, with $\sqrt{2}\beta \geq \beta_0 := (\nu^{-1} |\bar{u}(T)|^2_{L^\infty(\Omega, \mathbb{R})} + \lambda)^{1/2}$.

Hence we set $\beta = (\nu^{-1} |\bar{u}(T)|^2_{L^\infty(\Omega, \mathbb{R})} + \lambda)^{1/2}$ and solve \((\text{[42]})\), i.e.
\begin{equation}
P(-X(T) + \lambda/2I) + (X(T) + \lambda/2I)^\top P - PM^{-1}P + \nu S = 0,
\end{equation}
providing the initial guess $Y_0 = \beta M_c$. Let us denote the solution by $P_T^{[1]}$.

Remark 4.3. Notice that, proceeding as in the beginning of section \(3.1\) we see that a weak solution $w$ for $w_t = \nu \Delta w - \partial_x (\bar{u}(T) w) + (\lambda/2) w - \beta^2 w$, will satisfy the estimate
\begin{equation}
\frac{d}{dt} |w|^2_H \leq -\nu |\nabla w|^2_H + \nu^{-1} |\bar{u}(T)|^2_{L^\infty(\Omega, \mathbb{R})} |w|^2_H + \lambda |w|^2_H - 2\beta^2 |w|^2_H.
\end{equation}
That is, the lower bound $\beta_0$ works for the continuous system. However, when taking $\beta$ strictly bigger that $\beta_0$ we may expect that the stability is preserved for the discretized system, if $N_x$ is big enough.

(ii) We set $M = +\infty$ and the true $\chi$. In this case we can see that $R = (M_c D_x M^{-1})^\top$.

In some cases it may happen that $P_T^{[1]}$ is not a “good” initial guess. In some cases (as we have observed in some simulations) the step from $(+\infty, 1_\Omega)$ to $(+\infty, 0)$ seems to be too big, in other words $P_T^{[1]}$ is too far from the solution corresponding to $R = (M_c D_x M^{-1})^\top$. Having this in mind we connect the operators $I$ and $D_x$ by
the homotopy $H_{\tau} = (1 - \tau)I + \tau D$, $\tau \in [0, 1]$ and set $H := (M_{\tau}H_{\tau}M^{-1})^T$. Now let us fix $N_H \in \mathbb{N}_0$ and set the homotopy step $w = 1/N_H$ and solve
\[
P(-X(T) + \lambda/2I) + (-X(T) + \lambda/2I)^TP - P\rho H\rho^TP + \nu S = 0,
\]
providing the initial guess $Y_0 = P^{[1]}_T$. Let us denote the solution by $P^{[1+\rho]}_T$. Recursively we solve, for $l \in 2, \ldots, N_H$,
\[
P(-X(T) + \lambda/2I) + (-X(T) + \lambda/2I)^TP - P\rho H\rho^TP + \nu S = 0,
\]
providing the initial guess $Y_0 = P^{[1+(l-1)\rho]}_T$, and denote the solution by $P^{[1+l\rho]}_T$. After $N_H$ steps we have found a solution $P^{[2]}_T$ for
\[
P(-X(T) + \lambda/2I) + (-X(T) + \lambda/2I)^TP - P\rho H\rho^TP + \nu S = 0
\]
with $H_1 = (M_{\tau}D_{\tau}M^{-1})^T$.

(iii) We set the true $M$ and the true $\chi$. In this case $R$ is given by $\chi$1. Analogously to step (ii) we consider the homotopy $H_{\tau} = (1 - \tau)D + \tau P_M D_{\tau}$, set $H := (M_{\tau}H_{\tau}M^{-1})^T$ and, starting with $P^{[2]}_T$ we find, recursively after $N_H$ steps, a solution $P^{[3]}_T$ for
\[
P(-X(T) + \lambda/2I) + (-X(T) + \lambda/2I)^TP - P\rho H\rho^TP + \nu S = 0
\]
with $H_1 = (M_{\tau}P_M D_{\tau}M^{-1})^T = R$. That is $P^{[3]}_T$ solves $[12]$.

Of course, the number of homotopy steps $N_H$ may be taken different in steps (ii) and (iii). To get the convergence of the Newton method used to solve the algebraic Riccati equation at each homotopy step we may need, depending on the situations, to increase the number of homotopy steps $N_H$.

Notice, however that in step (iii) increasing $N_H$ can be sufficient for convergence at each homotopy step only if $M$ is big enough. Indeed we can see that the algebraic Riccati equation will have a solution up to the homotopy step before the last, because from the observability inequality $\chi_0$ we can also derive $\chi_0(0, T) \leq (1 - \tau)^{-2}C(1 - \tau)\chi_0^2((0, T), H)$, and from
\[
(1 - \tau)\chi_0^2((0, T), H)
= (1 - \tau)(1 - P_M\chi_0^2((0, T), L^2(\Omega, \mathbb{R})) + \chi_0^2((0, T), L^2(\Omega, \mathbb{R}))
\leq (1 - \tau)(1 - P_M\chi_0^2((0, T), L^2(\Omega, \mathbb{R})) + \chi_0^2((0, T), L^2(\Omega, \mathbb{R}))
\leq (1 - \tau)(1 - \tau)\chi_0^2((0, T), L^2(\Omega, \mathbb{R}))
\]
we arrive at $\chi_0(0, T) \leq (1 - \tau)^{-2}C(1 - \tau)\chi_0(0, T, H)$. Then, from this observability inequality it will follow that there exists a stabilizing control (for system $[12]$) of the form $\zeta(t) = (1 - \tau)\chi(t) + \tau \chi_0^2 P_M\chi_0(t)$, for $\tau < 1$. For the last homotopy step, that is, for $\tau = 1$ the observability is only known to hold if $M$ is big enough, so we cannot guarantee the existence of a stabilizing control of the form $\zeta(t) = \chi_0^2 P_M\chi_0(t)$ (for arbitrary $M$), and then neither a solution for the algebraic Riccati equation $[12]$. In step (iii), increasing $N_H$ should be sufficient to get the convergence at each homotopy step, because reasoning as above we can conclude that there exists a stabilizing control of the form $\zeta(t) = (1 - \tau)\eta(t) + \tau \chi_0 \eta(t)$, for all $\tau \in [0, 1]$.

Therefore, if convergence is not reached at a homotopy step in (ii) or at a homotopy step before the last in (iii) we probably need either more homotopy steps or to refine our mesh; if convergence is not reached only at the last homotopy step in (iii), then probably the number of controls is not enough.
In the simulations we present here, we have taken no more than \( N_h = 20 \) in the second step and no more than \( N_h = 10 \) in the third step. Notice however that increasing the number of homotopy steps does not mean that the computational time will be much bigger because the Newton method will converge faster at each homotopy step.

Finally, in the process of solving the differential Riccati equation, to find \( P^j \) solving (43) we provide the natural initial guess \( P^{j+1} \). Again, we cannot guarantee that the solution will always exist. If this process fails at some \( j \)-step, we can try to refine the mesh (in particular, by increasing the number \( N_t \) of time steps in (38)); if that does not work it probably means that the number of controls \( M \) is not sufficient.

4.4. Solving the discretized Oseen–Burgers system. Once we have constructed \( P \) we can simulate the evolution of the system (12); we look for \( \bar{\nu}(t) := \nu(t, \cdot) \) that solves the system

\[
\frac{\partial \bar{v}}{\partial t} + \nu \bar{M}^{-1} \bar{S} \bar{v} + \bar{M}^{-1} \bar{B} \partial_{\bar{u}} \bar{v} + \bar{R}^T \bar{R} P(\bar{v}) = 0, \quad \bar{v}(0) = \bar{v}_0,
\]

and expect \( \bar{v} \) to go exponentially to 0 as time increases, with an a priori prescribed rate \((\lambda/2) > 0 \) (cf. (14)), as time goes to infinity; recall that \( P \) depends on \( \lambda \) (cf. (41)). Notice that from (38), (36), and \( P = e^{\lambda t} Q_d \), it follows \( \bar{F} = \bar{R}^T \bar{R} P \bar{v} \).

Again, we will approximate \( \bar{v}(t) \approx [\bar{v}(j k)] \), \( j \in \{0, 1, \ldots, N_t\} \), and we apply a Crank-Nicolson inspired algorithm to solve system (44). For simplicity we denote \( \bar{F} := \bar{R}^T \bar{R} P \), for \( j \in \{0, 1, \ldots, N_t\} \). Set \( \bar{v}^j := \bar{v}(j k) = \bar{v}_0 \); then the idea is to construct, recursively, \( \bar{v}^{j+1} := \bar{v}((j + 1) k) \) from \( \bar{v}^j := \bar{v}(j k) \) by the scheme

\[
2/k(\bar{v}^{j+1} - \bar{v}^j) = -\nu \bar{M}^{-1} \bar{S} (\bar{v}^j + \bar{v}^{j+1}) - (\bar{M}^{-1} \bar{B} \partial_{\bar{u}} \bar{v}^j + \bar{B} \partial_{\bar{u}} \bar{v}^{j+1}) - (\bar{F}^j \bar{v}^j + \bar{F}^{j+1} \bar{v}^{j+1}),
\]

with \( \bar{u}^j := \bar{u}(j k) \), \( j \in \{0, 1, \ldots, N_t\} \). Then, working a little the above scheme, we can obtain

\[
\bar{v}^{j+1} = A_{\bar{\omega}}^{-1} A_{\bar{\omega}} \bar{v}^j - (k/2) A_{\bar{\omega}}^{-1} (\bar{B} \partial_{\bar{u}} \bar{v}^j + \bar{B} \partial_{\bar{u}} \bar{v}^{j+1}) - (k/2) A_{\bar{\omega}}^{-1} \bar{M} \left( \bar{F}^j \bar{v}^j + \bar{F}^{j+1} \bar{v}^{j+1} \right),
\]

with \( A_{\bar{\omega}} := \bar{M} + (k/2) \nu \bar{S} \) and \( A_{\bar{\omega}} := \bar{M} - (k/2) \nu \bar{S} \). Notice that the unknown \( \bar{v}^{j+1} \) is still present on the right hand side of (45). In the argument of the feedback operator we will replace \( \bar{v}^{j+1} \) by a preliminary guess \( \bar{v}^{G+1} \), and approximate \( \bar{B} \partial_{\bar{u}} \bar{v}^j + \bar{B} \partial_{\bar{u}} \bar{v}^{j+1} \) by \( \bar{B} \partial_{\bar{u}} \bar{v}^j + k(\bar{B} \partial_{\bar{u}} \bar{v}^j - \bar{B} \partial_{\bar{u}} \bar{v}^{j-1}) \) (where we define \( \bar{v}^{-1} := \bar{v}^0 = \bar{v}_0 \)). In this way we arrive at the scheme

\[
\bar{v}^{j+1} = A_{\bar{\omega}}^{-1} A_{\bar{\omega}} \bar{v}^j - k A_{\bar{\omega}}^{-1} ((1 + k/2) \bar{B} \partial_{\bar{u}} \bar{v}^j - (k/2) \bar{B} \partial_{\bar{u}} \bar{v}^{j-1}) - (k/2) A_{\bar{\omega}}^{-1} \bar{M} \left( \bar{F}^j \bar{v}^j + \bar{F}^{j+1} \bar{v}^{j+1} \right).
\]

We set \( \bar{v}^{G+1} \) as the “uncontrolled” output

\[
\bar{v}^{G+1} := A_{\bar{\omega}}^{-1} A_{\bar{\omega}} \bar{v}^j - k A_{\bar{\omega}}^{-1} ((1 + k/2) \bar{B} \partial_{\bar{u}} \bar{v}^j - (k/2) \bar{B} \partial_{\bar{u}} \bar{v}^{j-1}).
\]

4.5. Solving the discretized Burgers system. Concerning the evolution of the system (11). We look for \( \bar{u}(t) := u(t, \cdot) \) that solves the system

\[
\frac{\partial \bar{u}}{\partial t} + \nu \bar{M}^{-1} \bar{S} \bar{u} + (k/2) \bar{M}^{-1} \bar{B} \partial_{\bar{u}} \bar{u} + \bar{R}^T \bar{R} P(\bar{u} - \bar{u}) = 0, \quad \bar{u}(0) = \bar{u}_0,
\]

and expect \( \bar{u} \) to go exponentially to \( \bar{u} \), with an a priori prescribed rate \((\lambda/2) > 0 \), as time increases (with \( \bar{R} \) as in (36)). However this would be meaningful if \( \bar{u} \) were a solution
for the uncontrolled discrete system, which is not true. The solution of the uncontrolled
discrete system
\begin{equation}
(48) \quad \partial_t \vec{u}_S + \nu M^{-1} S \vec{u}_S + \frac{1}{2} M^{-1} B D_{\vec{u}_S} \vec{u}_S + \vec{h} = 0, \quad \vec{u}_S(0) = \vec{u}_0.
\end{equation}
will be an approximation \( \vec{u}_S \) of \( \vec{u} \). There is no reason to expect that \( e^{(\nu/n) t} (\vec{u} - \vec{u}_0) \) will remain bounded for \( t \in \mathbb{R}_0 \).

Nevertheless there is a way to check the rate of exponential stabilization \( \lambda \). We will just have to compute the discrete (fictitious) external force \( \vec{h}_I \), that makes \( \vec{u} \) a solution of the discrete system, that is,
\begin{equation}
(49) \quad \partial_t \vec{u} + \nu M^{-1} S \vec{u} + \frac{1}{2} M^{-1} B D_{\vec{u}} \vec{u} + \vec{h}_I = 0, \quad \vec{u}(0) = \vec{u}_0.
\end{equation}
Before that we present the scheme we apply. Suppose for the moment, that we know \( \vec{h}_I \).
Then, we follow the idea in section 4.3 and arrive at the scheme
\begin{equation}
(50) \quad \vec{u}_G^{j+1} = A_{\oplus}^{-1} A_{\ominus} \vec{u}^j - (k/2) A_{\ominus}^{-1} \left( (1 + k/2) B D_{\vec{u}} \vec{u}^j - (k/2) B D_{\vec{u}^{-1}} \vec{u}^{j-1} \right)
- (k/2) A_{\ominus}^{-1} M \left( \vec{h}_I^j + \vec{h}_I^{j-1} + \vec{f}(\vec{u}^j - \vec{u}_G^j) + \vec{f}^{j+1} (\vec{u}_G^{j+1} - \vec{u}^{j+1}) \right),
\end{equation}
with the preliminary “uncontrolled” guess \( \vec{u}_G^{j+1} \) given by
\begin{equation}
\vec{u}_G^{j+1} := A_{\oplus}^{-1} A_{\ominus} \vec{u}^j - (k/2) A_{\ominus}^{-1} \left( (1 + k/2) B D_{\vec{u}} \vec{u}^j - (k/2) B D_{\vec{u}^{-1}} \vec{u}^{j-1} \right)
- (k/2) A_{\ominus}^{-1} M \left( \vec{h}_I^j + \vec{h}_I^{j+1} \right).
\end{equation}

It remains to explain how we construct the force \( \vec{h}_I \). Actually, from our scheme
we can deduce that we only need to know the terms \( (k/2) A_{\ominus}^{-1} M \left( \vec{h}_I^j + \vec{h}_I^{j+1} \right) \), for \( j \in \{0, 1, \ldots, N_I - 1 \} \), that we can easily compute as
\begin{equation}
(k/2) A_{\ominus}^{-1} M \left( \vec{h}_I^j + \vec{h}_I^{j+1} \right)
= -\vec{u}^{j+1} + A_{\ominus}^{-1} A_{\ominus} \vec{u}^j - (k/2) A_{\ominus}^{-1} \left( (1 + k/2) B D_{\vec{u}} \vec{u}^j - (k/2) B D_{\vec{u}^{-1}} \vec{u}^{j-1} \right),
\end{equation}
(where we define \( \vec{u}^{-1} := \vec{u}_0 \)).

5. Numerical examples: the linear Oseen–Burgers system

Here we present some results of the numerical simulations we have performed concerning
the stabilization of system (12) to zero. Below, \( v_u \) stands for the solution of the uncontrolled (discretized) system (i.e., \( \zeta = 0 \)), and \( v \) (or \( v_\lambda \)) stands for the solution of the (discretized) system under the action of a (discretized) feedback controller
\( \zeta = \chi \eta, \) with \( \eta = e^{-\lambda t} E_0^O P_M^O((\chi Q_0^{L,\lambda})|_O) \) as in (15). If nothing is said in contrary \( O = (\inf\{\Omega \cap \text{supp} \chi\}, \sup\{\Omega \cap \text{supp} \chi\}) \).

5.1. Testing with a family of reference trajectories. We set \( \nu = 1/10, O := (3/2, 5/2), \)
\( \lambda = 2, \) and
\begin{equation}
(51) \quad \chi(x) = E_0^O (\sin((x - 3/2)\pi)|_O).
\end{equation}
That is, \( \chi = E_0^O \Sigma_1 \) (cf. section 2.1). Next we set the family of reference trajectories
\begin{equation}
(52) \quad \hat{u} = \hat{u}(x) = C_{uv} \sin(-t) \sin(ix) - \cos(3t) \sin((jx)),
\end{equation}
where the constant \( C_{uv} \) is chosen so that \( \|\hat{u}\|_{W^{1,\infty}} = 1 \). In this case we have that \( M_{\text{ref}} = \sqrt{120} > 10 \); so our question is if the number \( M \) of needed controls stay close to \( \sqrt{120} \).

We will test with the smaller number \( M = 4, \) and \( v_0(x) = \sin(2x) \). The function \( \chi \) and
the four controls are plotted in figure 2.
(a) The function $\chi$.  

(b) The first four sinus.

Figure 2. Basis for the control space $\{\chi E^O \eta \mid \eta \in \text{span}\{s_i \mid i \in \{1, 2, 3, 4\}\}\}$.  

In figure 3 we can check that the feedback control is able to stabilize the system with the desired rate. Then, we change the initial condition to $v_0(x) = \sin(x) - \sin(6x)$, and test for some other reference trajectories (with higher frequencies) in the family (52); in figure 4 we see that the feedback control is still able to stabilize the system with the desired rate; of course, the squared norm $|v|^2_{H}$ is to be understood as the discrete approximation $v^\top M v$ (cf. section 4.1).

Remark 5.1. There is no particular reason to test with $M$ strictly smaller than $M_{\text{ref}}$; trivially, if $M$ controls are enough to stabilize the system, then with $M_{\text{ref}} > M$ we can also stabilize the system.

5.1.1. Nonsmooth reference trajectory and nonsmooth initial condition. Again, we set $M = 4$ and $\lambda = 2$, $\nu = 1/10$ and $\chi$ as in (51). But, now we set $v_0(x) = 1_{[2, \pi]}(x)$ and the reference trajectory $\hat{u} = C_{n_1}1_{[0,1]}(\sin(-t)\sin(x) - \cos(3t)\sin(x))$, where

\begin{equation}
1_{[a, b]}(x) := \begin{cases} 
1, & \text{if } x \in [a, b] \\
0, & \text{if } x \in \Omega \setminus [a, b]
\end{cases}, \quad a, b \in \mathbb{R}
\end{equation}

and $C_{n_1}$ is taken so that $|\hat{u}|_{W^{1k}} = 1$.  

Figure 3. Convergence rate is achieved with the feedback control.
We can see in figure 5 that also in this case the feedback controller still stabilizes the system (12) to zero; in figure 6 we can see in particular that, in contrast to the smooth functions in the family (52), now $\hat{u}$ is not continuous (in the space variable); moreover $\hat{u}$ has a support disjoint from that of the control.

Remark 5.2. Notice that it makes sense to stabilize system (12) to zero, even if the “trajectory” $\hat{u} \in W^{wk}$ is not a weak solution for system (1).

5.2. Piecewise constant controls. Above we consider controls of the form $\chi_{E_{O}P_{M}}\eta$, following the results in [BRS11]. However, even if the results of the numerical simulations tell us that this controls seem indeed to behave quite well, we do not claim they are the best choice; analogous results could be (perhaps) derived for other type of controls. Taking this into consideration we present here also some results of simulations with other type of controls, namely for the case of piecewise constant controls (in space), these controls could be easier to realize in applications.

5.2.1. Stabilizing the heat system with just one control. We take $\hat{u} = 0$ in system (12). We recall that it is possible to find a control that is able to stabilize exponentially the
heat system to zero with any given rate $\lambda > 0$. Indeed let $\Omega = (0, \pi)$, $O = \Omega$, and $\chi(x) = 1_{[\pi/2, \pi/2]}(x)$ be defined as in (53).

We find that $(\chi, \xi_i)_H = \sqrt{2/\pi} \cos(3i/2) - \cos(5i/2)$, and since $i(5/2 \pm 3/2) \neq 2k\pi$ for all $(i, k) \in \mathbb{N}_0 \times \mathbb{N}$ we can conclude that $\chi_i := (\chi, \xi_i)_H \neq 0$, for all $i \in \mathbb{N}_0$.

Next let $\lambda > 0$ and consider the heat system controlled by a control of the form $\zeta = c\chi$ (with $c = c(t)$) and projected onto $F^\lambda := \text{span}\{\xi_i \mid \alpha_i < \lambda\}$, that we can rewrite as

\[(\partial_t / u) y^\lambda = \nu D^\lambda y^\lambda - c\chi^\lambda,\]

where $y^\lambda$ and $\chi^\lambda$ are the orthogonal projections of $y$ and $\chi$ onto $F^\lambda$, respectively and $D^\lambda$ is the diagonal matrix $D^\lambda = -\text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_J)$, where $\alpha_J$ is the biggest eigenvalue that is strictly less than $\nu^{-1}\lambda$. Next we prove that this finite-dimensional system is controllable to zero. For that it is enough to prove that the matrix $W = \begin{bmatrix} \chi^\lambda & \nu D^\lambda \chi^\lambda & (\nu D^\lambda)^2 \chi^\lambda & \ldots & (\nu D^\lambda)^{J-1} \chi^\lambda \end{bmatrix}$ has rank $J$ (see, e.g., [AS04, Theorem 3.3]). Notice that we can write $W = \text{diag}(\chi^\lambda)V$, where $V$ is a Vandermonde matrix and since the eigenvalues are all simple and since $\chi^\lambda$ has no vanishing coordinate, we see that $\det(W) = (\Pi_{j=1}^J \chi_j) \prod_{i<j}^{\nu\alpha_i - \nu\alpha_i} \neq 0$ (cf. [Pra94] chapter I, section 1.2], and [Shi77] chapter 1, Example c. in section 1.5]).

Therefore, there is a function $c = c^0(t)$ such that $c^0(t) \chi^\lambda$ drives the system (54) to zero, say at time $t = T > 0$. If we apply the control $c^0(t) \chi$ to the heat system the corresponding solution $v$ can be rewritten as $v = y^\lambda + z$ with $y^\lambda$ solving (54) and $z$ taking its values in the orthogonal space, in $H$, to $F^\lambda$. Then if we simply switch the control off, the system will go to zero with rate $\nu\alpha_{J+1} \geq \lambda$, for time $t \geq T$ (cf. [Bar11 Corollary 2.1]).

**Remark 5.3.** Notice that in the case $\hat{u} \neq 0$, it is not trivial (if possible) to prove that we can drive the projection $y^\lambda = P_{F^\lambda} v$ of the solution $v$ of system (12) to zero at a given time $t = T > 0$, with a control of the form $\zeta = c\chi$. Moreover, even if we manage to do it, we cannot simply switch the control off, because the term $\partial_x (\hat{u} v)$ will transfer energy to the space $F^\lambda$ and, in principle, the desired rate cannot be guaranteed anymore.

**Remark 5.4.** Notice that we can rewrite $c(t) \chi = \chi c(t) \chi$ and we can look for $c(t)$ as $c(t) = P_1 \eta := \eta(0) \chi |_{[\pi/3]}$. So we can just consider controls of the form $\chi P_1 \eta$, and we are in a similar setting as above just replacing $E^O_0 P_3^O (\eta|_O)$ by $P_1 \eta$.

In figure we give the result of the simulation with $\nu = 1/10$ and initial condition $v_0(x) = \sin(x)$; where $v_1$ and $v_1$ stand for the solutions corresponding to the rates of convergence 1.
and 4, respectively. We confirm that the piecewise constant control stabilizes the system with the prescribed rate.

![Graph](image)

(a) The function \( \chi \).

(b) Convergence rates of \( v_u \), \( v_1 \), and \( v_4 \).

**Figure 7.** The piecewise constant feedback control stabilizes the heat system with rates 1 and 4.

In figure 8 it is shown the controls \( \eta = c(t) \chi \) associated to the solutions in figure 7. Though it is possible to stabilize the system, we see that the magnitude of the control increases quite fast with the required rate of convergence. This shows that besides the dimension of the controller, also its cost could be relevant for applications.

![Graph](image)

(a) Control corresponding to the rate 1.

(b) Control corresponding to the rate 4.

**Figure 8.** The magnitude of the control increases with the required rate.

**Remark 5.5.** Notice that the eigenvalues (of \( M^{-1} S \approx -\Delta \)) remain simple for the discrete system. Indeed (cf. [Bof10, section 2]) the system of discrete eigenfunctions are the vectors in \( \left\{ \sin(m \lambda) \mid m \in \{1, 2, \ldots, N_x - 1\} \right\} \) and the corresponding system of eigenvalues are given by \( \left\{ (6/h^2)(1-\cos(mh)/2+\cos(mh)) \mid m \in \{1, 2, \ldots, N_x - 1\} \right\} \) (with \( \Omega = (0, \pi) \) and the mesh \( \Omega = (h, 2h, \ldots, (N_x - 1)h), 1 < N_x = \pi/h \in \mathbb{N}_0 \)). Notice that \( (d/dy)(1-\cos(yh)/2+\cos(yh)) = 3h \sin(yh)/((2+\cos(yh))^2) > 0 \), if \( y \in (0, N_x) \).
5.2.2. The case of a nonzero reference trajectory. Here we show that the piecewise constant control $\chi$, in section 5.2.1, can also stabilize system (12) to zero, in the case of a nonzero reference trajectory $\bar{u}$. As in section 5.2.1, we set $\nu = 1/10$ and $v_0(x) = \sin(x)$. But, now we take
\begin{equation}
\hat{u}(t, x) = t(t - 2)^2 \sin(2x) - \cos(2t) \sin(3x)
\end{equation}
and $\chi(x)$ as in (51). Figure 9 shows that the feedback piecewise constant control $\chi$ is still able to stabilize the system to zero. Of course, we cannot say for which subset of pairs $(\hat{u}, \lambda)$, of trajectories and rates, will the controller work; the example only shows that it makes sense to take also this class of piecewise constant controls under consideration.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure9a.png}
\includegraphics[width=0.5\textwidth]{figure9b.png}
\caption{(a) Convergence rates of $u_n$ and $v$. (b) Control corresponding to the rate 4.}
\end{figure}

5.3. Increasing number of needed controls. We set $\hat{u} = 0$, $\Omega = (0, \pi)$. In this example we show that for any given $n \in \mathbb{N}_0$ we can construct $\chi$ supported in a subset $\omega \subset \varpi \subset \Omega$, $\lambda > 0$, and an initial condition $v_0$, such that the first $2n$ controls cannot stabilize the system (12) to zero with the rate $\lambda$. However, by increasing the number of controls we can obtain the desired stabilization. Notice that here we look for $\chi$ with supp($\chi$) \not\subset \varpi (\text{cf. sections 3.3, 5.2.1 and 5.2.2}).

Let $n \in \mathbb{N}_0$. Set $v_0(x) = \sin((2n + 5)x)$, $\mathcal{O} = (2\pi/2n+5, (2n+3)\pi/2n+5)$, and $\chi = \chi_0^1 v_0^2 |_{\mathcal{O}}$. We claim that the controls $\chi P_{2n, \eta}^0$ cannot stabilize the equation with rate $\lambda > 2\nu(2n+5)^2$. Indeed, we can write $(v_0, \chi P_{2n, \eta}^0) = \sum_{i=1}^{2n} \eta_i \int_{\mathcal{O}} v_0^2 \phi_i \, d\mathcal{O} = \sum_{i=1}^{2n} \eta_i (l/2)^{3/2} \int_{\mathcal{O}} v_0^2 s_{2n+1} \phi_i \, d\mathcal{O}$ and $\int_{\mathcal{O}} \mathcal{S}_{2n+1} \phi_i \, d\mathcal{O} = 1/4 \int_{\mathcal{O}} (1 - \cos((2n+1) x) / (2n+1 - i - (2n+1) x) \, d\mathcal{O}$, with

$$
\mathcal{S}_j(x) := \sqrt{2} l \cos(i \pi / l), \quad x \in \mathcal{O}, \quad l := \text{length}(\mathcal{O}) = (2n+1)\pi/2n+5
$$

(cf. definition of the functions $s_n$ in section 2.1). Now, notice that since $i \leq 2n$, we have that $0 < 2n + 1 \pm i < 2(2n+1)$, thus we can conclude that $(v_0, \chi P_{2n, \eta}^0) = 0$. Therefore since the eigenspace $\text{span}\{v_0\}$ is preserved by the Laplacian we can conclude that the control cannot change the dynamics on this space. Thus, we conclude that the rate of convergence is at most $2\nu(2n+5)^2$.

Now we set $\nu = 1/10$; from above we know that for $n \in \{1, 2\}$ the rate of convergence $\lambda = 20 > 81^{1/5}$ is not achieved with the first $2n$ controls. Simulations below show that, in these examples, it is enough to add one more control to achieve the rate. In particular, we have $M = 2n + 1 \leq 5 < (\nu^{-1}\lambda)^{1/2} = 10\sqrt{2} = M_{\text{ref}}$.\[\]
In Figures 10 and 11 we see the results of the simulations for the cases $n = 1$ and $n = 2$; we can check the stabilization rate to zero of the heat system (i.e., system (12) with $\hat{u} = 0$).

(a) The function $\chi$.

(b) Convergence rates of $v_u$ and $v$.

**Figure 10.** Case $n = 1$. The first three controls can stabilize the heat system.

(a) The function $\chi$.

(b) Convergence rates of $v_u$ and $v$.

**Figure 11.** Case $n = 2$. The first five controls can stabilize the heat system.

6. **Numerical examples: the Burgers system**

It remains to confirm that the feedback control stabilizes the system (1)–(2) to a given reference trajectory $\hat{u}$, provided that $|u_0 - \hat{u}(0)|_H^2$ is “small”. We recall that $\hat{u}$ solves (1) with $\zeta = 0$ and $\hat{u}(0) = \hat{u}_0$. Below, we denote $d := u - \hat{u}$ and $d_u := u_u - \hat{u}$, where $u_u$ solves system (1)–(2) with $\zeta = 0$, and $u$ solves system (1)–(2) with the feedback control $\zeta$, as in (15), computed to stabilize the system (12) to zero.

6.1. **Local nature of the results and nonlinear nature of the equation.** As in section 5.1 we set $\nu = 1/10$, $\chi$ as defined in (51), $\lambda = 2$ and the trajectory $\hat{u} = C_{nr} (\sin(-t) \sin(8x) - \cos(3t) \sin(8x))$ from the family (52). Again we set $M = 4 <
\[ M_{\text{ref}} = \sqrt{120}. \] Next, we consider the family of initial conditions \( u_0 = u_0^\delta := d_0^\delta + \hat{u}_0 \) with \( d_0 = d_0^\delta = \delta (\sin(x) - \sin(6x)) \), and \( \delta \in \mathbb{R} \setminus \{0\} \).

In figure 12 we can see that the feedback control is able to stabilize, with the desired rate, the nonlinear system (1)–(2) to the trajectory \( \hat{u} \), provided that \( d_0 \) is small enough. We can see that, for \( |\delta|_R > 1 \), the stabilization rate is not guaranteed; while for \( |\delta| \leq 1 \) it holds. For example, for \( |\delta|_R \leq 1 \) we can see that the local maxima of the plotted curves seem either to converge to a real number or to decrease, while for \( |\delta|_R > 1 \) those local maxima seem to go to infinity. Notice that the radius 1 here is suitable for this example, for other settings the stabilization may hold only for smaller \( |\delta|_R \).

In figure 13 we can see that the uncontrolled systems does not go exponentially to \( \hat{u} \) (at least not with the rate \( \lambda = 2 \)); here we have plotted the curves corresponding to some of those values of \( \delta \) in figure 12 (for the other the behavior is similar).

We can also see the nonlinear nature of the equations, because the behavior changes with the sign of the initial condition.

**Remark 6.1.** The results correspond to the case in which we take a fictitious external force \( h_f \) (i.e., an approximation of \( h \)) that makes \( \hat{u} \) a solution of the discrete system (cf. section 4.5).

**Figure 12.** Convergence rate to \( \hat{u} \) holds locally.

**Figure 13.** Uncontrolled case.
6.2. Real versus fictitious external force behavior. Here we are in the same setting as in section 6.1. But, now we fix $\delta = 1$, and consider $\hat{u}$ in the longer time interval $t \in [0, 10]$. We compare the numerical results in the case when we take the real external force $h = -\partial_t \hat{u} - \hat{u} \partial_x \hat{u} + \nu \Delta \hat{u}$ with those in the case when we take the fictitious external force $h_f$ (cf. Remark 6.1). We denote $d = u - \hat{u}$ and $d_r = u_r - \hat{u}$, where $u_r$ solves (1)–(2) (that is, with the real external force $h$), and $u$ solves (1)–(2) with $h_f$ in the place of $h$. In both cases $\zeta$ is the feedback control as in (15). In figure 14, we confirm the rate of convergence of $u$ to $\hat{u}$ in the entire time interval, while for $d_r$ the rate is confirmed until time $t = 6$; after time $t = 6$ we see that $d_r$ remains bounded, this just means that the magnitude of $u_r - \hat{u}$ has reached that of the discretization error of our solver, and consequently we cannot expect the magnitude of $u_r - \hat{u}$ to decrease more.

![Figure 14. Fictitious versus real external force.](image)

6.3. On the discretization error. Here we are in the same setting as in section 6.1 with $\delta = 1$. We observe that, following the scheme (50), with the exact external force $h$ (in the place of $h_f$), the discrete solution $\hat{u}_S$ for (48) will be close to $\hat{u}$. Moreover, $\hat{u}_S$ converges to $\hat{u}$ as $(k, h) \to (0, 0)$. We would like to say that here we just want to show that we can trust the results of the simulations; it is not our intention to compare our algorithm/discretization to solve the Burgers and Oseen-Burgers systems with existing ones. Concerning further numerical approaches for control of Burgers equation we refer the reader to [TBR10, KV99, KV02, KX05, HV02, BK84].

7. Final remarks

We have presented some estimates on the number of internal controls $M$ we need to exponentially stabilize the Burgers system to a given reference trajectory $\hat{u} = \hat{u}(t, x)$. In the case we impose no constraint on the support of the control we can derive a better estimate comparing with the constrained case (cf. sections 3.1 and 3.2), and we have presented the results of some numerical simulations that suggest that an estimate like that in the unconstrained case might hold also in the constrained one.

We have focused on the viscous 1D Burgers system. However, we are convinced that the challenge to find an estimate for $M$ will present analogous difficulties for the cases of 2D and 3D Burgers and Navier–Stokes systems, and also for a wide class of parabolic systems.

Our results do not apply to the case of nonviscous Burgers equation (i.e., to the case we take $\nu = 0$ in (1)), that is a completely different problem. We do not even know if a
finite number $M$ of controls is enough to stabilize the system (in a general situation the number $M$ of needed controls will go to $+\infty$ as $\nu$ goes to 0).

The value $\nu = 1/10$ we use in the simulations is (perhaps) too big for many applications. Of course we can take smaller $\nu$ but, in that case we may need to take also a finer mesh in order to guarantee that the stabilization observed for the discretized system in the numerical simulations, will also hold for the continuous system. Notice that, when the numerical solution for system (1) goes to $\hat{u}$ as time increases, we can extrapolate that also the evaluations $u(t, ih), i \in \{1, 2, \ldots, N_x - 1\}$, of the continuous solution at the spatial mesh points will go to $\hat{u}(t, ih)$ as time increases. Recall that if $|u(t) - \hat{u}(t)|_H$ goes to 0 as $t$ increases, then also $|u(t) - \hat{u}(t)|_V$ does (provided that $\hat{u} \in W^1$, due to the smoothing property of the system (8)). However, the fact that $|u(t, ih) - \hat{u}(t, ih)|_R$ goes to 0, for all $i \in \{1, 2, \ldots, N_x - 1\}$, as time increases, is in general not enough to conclude that $u$ goes to $\hat{u}$. Indeed, from [JT92, Theorem 4.2] (for the case of the Navier–Stokes system in a two-dimensional Torus) we can derive that to conclude that $u$ goes to $\hat{u}$, the space step $h$ should be taken proportional to $1/\nu^2(1 - 2 \log(\nu))$ (for small $\nu$); and supposing that a similar estimate holds for the 1D Burgers system, it would follow that the number $N_x$ of space points (determining nodes) should be proportional to $\nu^{-2}(1 - 2 \log(\nu))$. Notice that the computational effort and computational time will increase with $N_x$. We refer
also to [JT93] and [FMRT01] chapter III, section 2, and references therein, concerning the estimates on the number of determining nodes.

The mathematical theory concerning stabilization to time-dependent trajectories (cf. [BRS11]) is not so developed as for stabilization to a stationary state (cf. [RT10, BT11, Bar12, Bar11, BLT06, BT04, Rav07]). However, since these problems arise in applications, methods to solve these problems numerically have already been developed (see, e.g., [FK12, Kor06, Kor08] and references therein), notice that in this setting “trajectory” will often mean a suitable evolutionary discrete process $u_0 \in Z, u_{i+1} = S(u_i) \in Z, i \in \mathbb{N}$, where $Z$ is a Hilbert space. Other approaches can be found, for example, in [KMV09] (in particular, see section 4 concerning trajectory tracking) and in [Gun03] (in particular, see section 7.1 concerning linear feedback control of Navier–Stokes flows).

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REFERENCES


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