H. Albrecher, S. Thonhauser

Optimal dividend strategies for a risk process under force of interest
OPTIMAL DIVIDEND STRATEGIES FOR
A RISK PROCESS UNDER FORCE OF INTEREST*

HANSJÖRG ALBRECHER a,b  STEFAN THONHAUSER a

a Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences,
Altenbergerstrasse 69, A-4040 Linz, Austria
b Graz University of Technology, Steyrergasse 30, A-8010 Graz, Austria

Abstract
In the classical Cramér-Lundberg model in risk theory the problem of maximizing the expected cumulated discounted dividend payments until ruin is a widely discussed topic. In the most general case within that framework it is proved (Gerber (1969), Azcue & Muler (2005), Schmidli (2007)) that the optimal dividend strategy is of band type. In the present paper we discuss this maximization problem in a generalized setting including a constant force of interest in the risk model. The value function is identified in the set of viscosity solutions of the associated Hamilton-Jacobi-Bellman equation and the optimal dividend strategy in this risk model with interest is derived, which in the general case is again of band type and for exponential claim sizes collapses to a barrier strategy. Finally, an example is constructed for Erlang(2)-claim sizes, in which the bands for the optimal strategy are explicitly calculated.

1 Introduction

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\) be a filtered probability space on which all random processes and variables introduced in the sequel are defined. Consider the following stochastic model for the risk reserve process \(R = \{R_t\}_{t \geq 0}\) of an insurance portfolio

\[
R_t = x + ct - \sum_{k=1}^{N_t} Y_k + \int_0^t R_s ds.
\]

The number of claims \(N = \{N_t\}_{t \geq 0}\) is modelled as a homogeneous Poisson process with parameter \(\lambda\) which has the càdlàg property \((N_{t+} = N_t)\). The incoming premiums are assumed to be collected continuously over time at a constant rate \(c\). The claim amounts \(\{Y_k\}_{k \in \mathbb{N}}\) are an iid sequence of positive random variables with continuous distribution function \(F_Y\). The integral term represents the additional income resulting from the constant force of interest \(i > 0\) on the free surplus (see for instance Paulsen [9], where the existence of such a process \(R\) is proved). A similar model was dealt with in Albrecher et al. [2] and Paulsen & Gjessing [10, 11]. In this paper we are interested in identifying the optimal strategy to pay out dividends from process (1) to shareholders during the period of solvency.

Let \(L_t\) denote the accumulated paid dividends up to time \(t\). We call a dividend strategy \(L = \{L_t\}_{t \geq 0}\) admissible if it is an adapted càdlàg (previsible, \(L_{t-} = L_t\)) and non-decreasing process. Further we require \(L_{t+} - L_t \leq R^L_t\) such that paying dividends can not cause ruin, where the controlled process is defined via

\[
R^L_t = x + ct - \sum_{k=1}^{N_t} Y_k + i \int_0^t R^L_s ds - L_t.
\]

The càdlàg property of the reserve process and the càglàd property of the dividends process imply that \(R^L_{t+} - R^L_t\) is always due to a claim and \(R^L_{t+} - R^L_t\) is due to some singular dividend payment. Although not standard in the literature, this càglàd assumption for the dividends will simplify the analysis (and the previsibility of the control is then also ensured by the càglàd property).

The performance of an admissible strategy \(L\) is measured by the function

\[
V_L(x) = \mathbb{E}\left( e^{-\delta t} \int_0^t e^{-\delta s} dL_s \bigg| R^L_0 = x \right), \tag{2}
\]

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i.e. the expectation of the discounted dividend payments until the time of ruin $\tau^L = \inf\{t|R_t^L < 0\}$ of the controlled process. Here $\delta > 0$ denotes the discount factor, which can also be interpreted as a measure of the preference of shareholders to receive payments earlier rather than later during the lifetime of the risk process. The value function of the maximization problem is then given through

$$V(x) = \sup_{L \in \Pi} V_L(x),$$

where the supremum is taken over the set $\Pi$ of all admissible strategies.

Optimization problems of the form (3) are a classical topic in stochastic control theory (see for instance Schmidt [13] for a recent survey). Under the assumption that the underlying risk process $R$ is modelled by a Cramér-Lundberg process (i.e. (1) with $i = 0$), it was first shown in Gerber [7] by a discrete approximation and then a limiting argument that the optimal dividend strategy according to the criterion (2) is of so-called band type. This result was recently rederived by means of viscosity theory in Azcue and Muler [3]. It is a natural question to ask for an analogous result in the presence of an interest force $i > 0$ on the free surplus, not the least because, from a practical perspective, the use of a discount factor $\delta > 0$ for the dividends in the objective function acknowledging the time value of money should be complemented by such an effect for the underlying risk process, too. It is intuitively not surprising that the dividend maximization problem is only well formulated for $i < \delta$ (for exponential claims we will also demonstrate this fact explicitly in Section 4).

As in the classical Cramér-Lundberg case, one can not expect the value function (3) to be a classical solution of the associated Hamilton-Jacobi-Bellman (HJB) equation. Like Azcue & Muler [3] in the case $i = 0$, we therefore use the methodology of viscosity solutions to identify the optimal strategy for $i > 0$.

The outline of the paper is as follows. After establishing some basic properties of the value function (3), the corresponding HJB equation is derived and the value function is identified as a viscosity solution of this HJB equation (Proposition 2.3). Typically, dividend maximization problems in the Cramér-Lundberg setting lack an initial condition (cf. Azcue & Muler [3], Gerber [7], Schmidt [13], Mufi & Sulam [8]) circumvent this problem by considering a slightly different risk model that does provide an initial value for the maximization problem). Therefore we first prove uniqueness of the viscosity solution of the HJB equation for a given initial condition via a comparison principle (Proposition 2.4) and in a second step we show that every viscosity supersolution dominates the value function (Proposition 2.6). In that way we can characterize the value function as the viscosity supersolution with the smallest initial value fulfilling the same growth conditions.

The construction of the optimal strategy of band type needs some care concerning the behaviour of the value function at points where differentiability may not be fulfilled (Propositions 2.11 and 2.12, which also indicate already how to construct the optimal solution along the arguments of Schmidt [13]).

In Section 3 the existence and uniqueness of the solution to the integro-differential part of the HJB equation in the respective regions are established and properties of the crucial sets needed for the definition of the optimal strategy are derived. Eventually the appropriate band strategy is formulated and its optimality is proved (Proposition 3.3).

In Section 4.1 the case of exponentially distributed claim sizes is investigated in more detail and it is shown that in this case the optimal band strategy collapses to a barrier strategy, including a study of conditions on parameter values under which the optimal barrier is in fact in 0 (this complements results of Paulsen & Gjessing [10], who investigated optimal barrier values for the risk process (1) within the class of barrier strategies).

Finally, in Section 4.2 an example for Erlang(2)-distributed claims is identified for which the optimal band strategy can be explicitly calculated.

2 Value function and viscosity solutions

2.1 Basic properties of the value function

Let us first derive some bounds for the value function and its first derivative.
Proposition 2.1. For \( i < \delta \) we have
\[
x + \frac{c}{\delta + \lambda} \leq V(x) \leq \frac{c}{\delta - i} + x \max\{1, \frac{i}{\delta - i}\}.
\]

Proof. The controlled process
\[
R_t^L = x + ct - \sum_{k=1}^{N_t} Y_k + i \int_0^t R_s^L ds - L_t
\]
is clearly upper-bounded by
\[
R_t^L \leq e^{t} \left( x + c \int_0^t e^{-is} ds \right)
\]
and the growth rate in \( t \) of the right hand side is \( e^t (ix + c) \). We have the possibility of immediately paying an amount \( 0 \leq a \leq x \) and get
\[
V(x) \leq \max_{0 \leq a \leq x} \left( a + \int_0^\infty e^{-\delta s} e^{i x (i(x-a) + c) ds} \right) = \max_{0 \leq a \leq x} \left( a \frac{\delta - 2i}{\delta - i} + \frac{ix + c}{\delta - i} \right).
\]
The maximum is attained in either \( a = 0 \) or \( a = x \). Therefore we get the upper bound
\[
V(x) \leq \max \left\{ x + \frac{c}{\delta - i}, \frac{ix + c}{\delta - i} \right\}.
\]
On the other hand, we get a lower bound for \( V(x) \) when we pay the initial surplus \( x \) and all incoming premia immediately as dividends and the first claim that occurs (after an exponential time \( \tau_1 \)) causes ruin:
\[
V(x) \geq V_{L_0}(x) = x + c \mathbb{E} \left( \int_0^{\tau_1} e^{-\delta s} ds \right) = x + \frac{c}{\delta + \lambda}.
\]
\[\square\]

Proposition 2.2. For \( 0 \leq x < y \) we have the following inequalities
\[
y - x \leq V(y) - V(x) \leq V(x) \left( \frac{iy + c}{ix + c} \right)^{\frac{i + \lambda}{i}} - 1
\]

Proof. For \( \epsilon > 0 \) let \( L_\epsilon \) be an \( \epsilon \)-optimal strategy for initial capital \( x \) (i.e. \( V_{L_\epsilon}(x) \geq V(x) - \epsilon \)). For \( y > x \) define \( \tilde{L} \) such that an amount \( y - x \) is paid as dividend immediately followed by using the strategy \( L_\epsilon \). We have
\[
V(y) \geq y - x + V_{L_\epsilon}(x) \geq y - x + V(x) - \epsilon.
\]
Because this holds for all \( \epsilon > 0 \) we get
\[
V(y) - V(x) \geq y - x.
\]
For the other direction let \( 0 \leq x < y \) and \( \epsilon > 0 \). Define \( \tilde{L} \) for initial capital \( x \) as follows. Nothing is done as long as the reserve stays below \( y \) and then an \( \epsilon \)-optimal strategy \( L_\epsilon \) for initial capital \( y \) is applied. The reserve reaches \( y \) not before time \( t_0 = \frac{1}{\delta} \log \left( \frac{iy + c}{ix + c} \right) \) and it is further assumed that there is no payment at all if a claim occurs before \( t_0 \). Hence
\[
V(x) \geq V_{\tilde{L}}(x) \geq e^{-\delta \lambda t_0} V_{L_\epsilon}(y) \geq e^{-\delta \lambda t_0} (V(y) - \epsilon).
\]
Finally we arrive at
\[
V(y) - V(x) \leq V(x) \left( \frac{iy + c}{ix + c} \right)^{\frac{i + \lambda}{i}} - 1.
\]
\[\square\]

From the above and [15], we get that \( V(x) \) is increasing and locally Lipschitz on \([0, \infty)\) (apply a Taylor expansion to the upper bound around \( x \) to see this) which by Rademacher’s Theorem ensures the existence of the derivative almost everywhere and then \( 1 \leq V'(x) \leq \frac{c - \lambda}{x^2} V(x) \). Furthermore \( V(x) \) is Lipschitz on compact sets which implies that it is absolutely continuous.
2.2 Representation as a viscosity solution

The value function \(V(x)\) fulfills the dynamic programming principle for any stopping time \(\gamma\),

\[
V(x) = \sup_{L \in \Pi} \mathbb{E} \left( \int_0^{\tau_{x \land \gamma}} e^{-\delta s} dL_s + e^{-\delta (\tau_{x \land \gamma})} V(R^L_{\tau_{x \land \gamma}}) \right),
\]

which can be shown analogously to the proof of Proposition 3.1 of [3] (with \(x_{\text{max}}\) replaced by \(e^{i\gamma} (x + c \int_0^\gamma e^{-is} ds)\)).

Now let us define the operator

\[
\mathcal{L}_u(x) = (c + ix)u'(x) - (\delta + \lambda)u(x) + \lambda \int_0^x u(x - y)dF_V(y).
\]

Standard arguments from stochastic control (see [6]) imply the HJB equation

\[
\max \{1 - u'(x), \mathcal{L}_u(x)\} = 0.
\]

But, as mentioned in the introduction, we can not expect the value function to be a classical solution to (5). Therefore we need another concept of solutions for this type of equation. We choose the concept of viscosity solutions which is introduced in the following.

**Definition 2.1.** A function \(u : [0, \infty) \to \mathbb{R}\) is called a viscosity subsolution of (5) at \(x \in (0, \infty)\) if any continuously differentiable function \(\psi(x) : (0, \infty) \to \mathbb{R}\) with \(\psi(x) = u(x)\) such that \(u - \psi\) reaches a maximum at \(x\) satisfies

\[
\max \{1 - \psi'(x), \mathcal{L}_\psi(x)\} \geq 0.
\]

We say that a function \(\pi : [0, \infty) \to \mathbb{R}\) is a viscosity supersolution of (5) at \(x \in (0, \infty)\) if any continuously differentiable function \(\phi(x) : (0, \infty) \to \mathbb{R}\) with \(\phi(x) = \pi(x)\) such that \(\pi - \phi\) reaches a minimum at \(x\) satisfies

\[
\max \{1 - \phi'(x), \mathcal{L}_\phi(x)\} \leq 0.
\]

A function \(u(x) : [0, \infty) \to \mathbb{R}\) is a viscosity solution if it is both a viscosity sub- and supersolution.

**Remark 2.1.** At some points later on will also make use of a different but equivalent (Sahay [12], Benth et al. [4]) definition of a viscosity sub- and supersolution: Define the modified operator

\[
\mathcal{L}_{u, \psi}^*(x) = (c + ix)\psi'(x) - (\delta + \lambda)\psi(x) + \lambda \int_0^x \psi(x - y)dF_V(y).
\]

A function \(u : [0, \infty) \to \mathbb{R}\) is a viscosity subsolution of (5) at \(x \in (0, \infty)\) if any continuously differentiable function \(\psi(x) : (0, \infty) \to \mathbb{R}\) with \(\psi(x) = u(x)\) such that \(u - \psi\) reaches a maximum at \(x\) satisfies

\[
\max \{1 - \psi'(x), \mathcal{L}_{u, \psi}^*(x)\} \geq 0.
\]

A function \(\pi : [0, \infty) \to \mathbb{R}\) is a viscosity supersolution of (5) at \(x \in (0, \infty)\) if any continuously differentiable function \(\phi(x) : (0, \infty) \to \mathbb{R}\) with \(\phi(x) = \pi(x)\) such that \(\pi - \phi\) reaches a minimum at \(x\) satisfies

\[
\max \{1 - \phi'(x), \mathcal{L}_{\pi, \phi}^*(x)\} \leq 0.
\]

Later on we will need the following two properties of the derivatives of some test functions.

**Remark 2.2.** A continuously differentiable function \(\psi : (0, \infty) \to \mathbb{R}\) such that \(\pi - \psi\) reaches a maximum at \(x > 0\) with \(\psi'(x) = q\) exists if and only if

\[
\liminf_{y \uparrow x} \frac{\psi(y) - \psi(x)}{y - x} \geq q \geq \limsup_{y \downarrow x} \frac{\psi(y) - \psi(x)}{y - x}.
\]

A continuously differentiable function \(\phi : (0, \infty) \to \mathbb{R}\) such that \(\pi - \phi\) reaches a minimum at \(x > 0\) with \(\phi'(x) = q\) exists if and only if

\[
\liminf_{y \uparrow x} \frac{\pi(y) - \pi(x)}{y - x} \geq q \geq \limsup_{y \downarrow x} \frac{\pi(y) - \pi(x)}{y - x}.
\]
Remark 2.3. Note that for a continuously differentiable test function $\psi$ (as required in the definition of viscosity solutions) the operators $\mathcal{L}_\psi$ and $\mathcal{L}^L_{\nu, \psi}$ are continuous for $x \geq 0$, so that we do not have to work with the upper semi-continuity as in Mif\l{} and Sulem [8].

The next proposition characterizes the value function as a viscosity solution. The supersolution proof is in the spirit of [3], whereas the subsolution proof is related to the approach in [8].

Proposition 2.3. The value function $V$ is a viscosity solution of the HJB equation (5).

Proof. We start with showing that $V$ is a viscosity supersolution. Fix $l \geq 0$ and let $h > 0$ be small enough such that $e^{\lambda h} (x + (c - l) \int_0^h e^{-is} ds) \geq 0$. Let $\tau_1$ denote the time of the first claim occurrence. From the dynamic programming principle we derive

$$V(x) = \sup_{L \in \Pi} E \left( \int_0^{\tau_1 \wedge h} e^{-\delta s} dL_s + e^{-\delta (\tau_1 \wedge h)} V \left(R^L_{\tau_1 \wedge h}\right) \right)$$

$$\geq e^{-\lambda h} \int_0^h e^{-\delta l} ds + e^{-\delta \lambda h} V \left( e^{\lambda h} \left(x + (c - l) \int_0^h e^{-is} ds\right) \right)$$

$$+ \int_0^h \lambda e^{-\lambda t} \left[ \int_0^t e^{-\delta s} l ds + e^{-\delta t} \right] \int_0^t e^{\lambda t} \left(V \left( e^{\lambda t} \left(x + (c - l) \int_0^t e^{-is} ds\right) - y\right) dF_Y(y) \right) dt.$$ 

This further leads to

$$0 \geq 1 - e^{-\delta \lambda h} h^{-l} \left[ - \frac{V \left( e^{\lambda h} \left(x + (c - l) \int_0^h e^{-is} ds\right) \right) - V(x)}{h} - 1 - e^{-\delta \lambda h} \right]$$

$$+ \frac{\lambda}{h} \int_0^h e^{-\delta \lambda t} \left[ \int_0^t e^{\lambda t} \left(x + (c - l) \int_0^t e^{-is} ds\right) - y\right] dF_Y(Y) dt.$$ 

Now let $\phi$ be a continuously differentiable test function with $V(x) = \phi(x)$ and $V - \phi$ attaining a minimum in $x$. We get

$$0 \geq 1 - e^{-\delta \lambda h} h^{-l} \left[ - \frac{\phi \left( e^{\lambda h} \left(x + (c - l) \int_0^h e^{-is} ds\right) \right) - \phi(x)}{h} - 1 - e^{-\delta \lambda h} \right]$$

$$+ \frac{\lambda}{h} \int_0^h e^{-\delta \lambda t} \left[ \int_0^t e^{\lambda t} \left(x + (c - l) \int_0^t e^{-is} ds\right) - y\right] dF_Y(Y) dt.$$ 

Using Taylor expansion w.r.t. $h$ at $h = 0$ and neglecting second order terms, $e^{\lambda h} \left(x + (c - l) \int_0^h e^{-is} ds\right) \approx x + h(ix + (c - l))$, we get for $h \to 0$ and using continuity of $V$ and differentiability of $\phi$

$$0 \geq l(1 - \phi'(x)) + (ix + c)\phi'(x) - (\delta + \lambda) V(x) + \lambda \int_0^h V(x - y) dF_Y(y).$$

Inequality (8) holds for an arbitrary $l \geq 0$ (using a strategy $L_t = tl$). This gives $1 - \phi'(x) \leq 0$ and for $l = 0$ we get $\mathcal{L}^L_{\nu, \psi}^L(x) \leq 0$. Therefore we have that $V$ is a viscosity supersolution of (5).

Next we will identify the viscosity subsolution property using Definition 2.1. For some function $\psi \in C^1(0, \infty)$ fulfilling

$$0 = V(x_0) - \psi(x_0) > V(x) - \psi(x) \quad \forall x \neq x_0, x \in (0, \infty),$$

for some $x_0 \in (0, \infty)$, we have to show

$$\max \{ 1 - \psi(x_0), \mathcal{L}^L_{\nu, \psi}(x_0) \} \geq 0.$$
Assume the contrary. Because $\psi$, $\psi'$ and $V$ are continuous, the operator $L_\psi$ is continuous, too. Therefore some $r > 0$ and $\xi > 0$ exist with
\[ \max \{ 1 - \psi'(x), L_\psi(x) \} < -\delta \xi, \quad \forall x \in (x_0 - r, x_0 + r) = B, \]
and such that for $x' = x_0 \pm r$ we have
\[ V(x') \leq \psi(x') - \xi. \]

Further choose $r$ such that $B \subset (0, \infty)$. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence with $x_n \to x_0$ and without loss of generality assume $x_n \in B$ for all $n \in \mathbb{N}$. Because of the continuity of $\psi$ and $V$ we have $| V(x_n) - \psi(x_n) | \to 0$. From now on we look at the reserve with initial capital $x_n$ which is controlled by an arbitrary admissible strategy $L \in \Pi$, $R_{t}^{L,x_n} = \{ R_{t}^{L,x_n} \}_{t \geq 0}$. Define
\[ \tau_n = \inf \{ t > 0 \mid R_{t}^{L,x_n} \notin B \} \]
and denote by $\tau^* = \tau_n \wedge T$ for some $T > 0$. Look now at the set $\{ \tau^* \} \bigcup \{ t \}$ first, leaving $B$ before time $T$. We have, from the construction of the process, that either $x_0 + r$ is reached which implies $R_{\tau^*}^{L,x_n} = R_{\tau^*+}^{L,x_n} = x_0 + r$, or a jump happens leading to $R_{\tau^*}^{L,x_n} \geq R_{\tau^*}^{L,x_n}$ and $R_{\tau^*}^{L,x_n} \leq x_0 - r$. Since $V$ is increasing and also $\psi$ is increasing on $B$, we get from $\psi' > 1$,
\[ V(R_{\tau^*}^{L,x_n}) \leq V(x') \leq \psi(x') - \xi \leq \psi(R_{\tau^*}^{L,x_n}) - \xi. \]

On the set $\{ \tau^* = T \}$, $R_{\tau^*}^{L,x_n} \leq R_{\tau^*}^{L,x_n}$ gives
\[ V(R_{\tau^*}^{L,x_n}) \leq \psi(R_{\tau^*}^{L,x_n}). \]

Altogether
\[ e^{-\delta \tau^*} \psi(R_{\tau^*}^{L,x_n}) \leq e^{-\delta \tau^*} - \psi(R_{\tau^*}^{L,x_n}) = e^{-\delta \tau^*} \xi I(\tau_n = \tau^*). \]

Apply the Itô formula to $e^{-\delta \tau^*} - \psi(R_{\tau^*}^{L,x_n})$:
\[ e^{-\delta \tau^*} - \psi(R_{\tau^*}^{L,x_n}) - \psi(x) = \int_{0}^{\tau^*} e^{-\delta s} (c + i R_{s}^{L,x_n}) \psi'(R_{s}^{L,x_n}) \psi(R_{s}^{L,x_n}) \, ds - \delta \int_{0}^{\tau^*} \psi(R_{s}^{L,x_n}) e^{-\delta s} \, ds \]
\[ - \int_{0}^{\tau^*} \psi'(R_{s}^{L,x_n}) e^{-\delta s} \, dL_{s} + \sum_{R_{s}^{L,x_n} \neq R_{s}^{L,x_n} \wedge s \leq \tau^*} \left( \psi(R_{s}^{L,x_n}) - \psi(R_{s}^{L,x_n}) \right) e^{-\delta s} \]
\[ + \sum_{R_{s}^{L,x_n} \neq R_{s}^{L,x_n} \wedge s \leq \tau^*} \left( \psi(R_{s}^{L,x_n}) - \psi(R_{s}^{L,x_n}) \right) e^{-\delta s}. \]

Note that $R_{s}^{L,x_n} - R_{s}^{L,x_n} = -(L_{s} - L_{s})$ and therefore $\sum_{R_{s}^{L,x_n} \neq R_{s}^{L,x_n} \wedge s \leq \tau^*} \left( \psi(R_{s}^{L,x_n}) - \psi(R_{s}^{L,x_n}) \right) e^{-\delta s} = -\sum_{L_s \neq L_{s}} e^{-\delta s} \left( \int_{L_s}^{L_{s} - L_{s}} \psi'(R_{s}^{L,x_n} - u) \, du \right)$. Because $\psi'(x) > 1$ for $x \in B$ we get
\[- \left( \int_{0}^{\tau^*} \psi'(R_{s}^{L,x_n}) e^{-\delta s} \, dL_{s} + \sum_{L_{s} \neq L_{s} \wedge s \leq \tau^*} e^{-\delta s} \left( \int_{L_{s}}^{L_{s} - L_{s}} \psi'(R_{s}^{L,x_n} - u) \, du \right) \right) \leq \]
\[- \left( \int_{0}^{\tau^*} e^{-\delta s} \, dL_{s} + \sum_{L_{s} \neq L_{s} \wedge s \leq \tau^*} e^{-\delta s} (L_{s} - L_{s}) \right) = - \int_{0}^{\tau^*} e^{-\delta s} \, dL_{s}. \]

The last equality holds because the dividends process is left-continuous. Plugging this into (9) we obtain the inequality
\[ e^{-\delta \tau^*} - \psi(R_{\tau^*}^{L,x_n}) \leq \psi(x_n) - \int_{0}^{\tau^*} e^{-\delta s} \, dL_{s} + \int_{0}^{\tau^*} e^{-\delta s} \left( c + i R_{s}^{L,x_n} \psi'(R_{s}^{L,x_n}) - \delta \psi(R_{s}^{L,x_n}) \right) \, ds \]
\[ + \sum_{R_{s}^{L,x_n} \neq R_{s}^{L,x_n} \wedge s \leq \tau^*} \left( \psi(R_{s}^{L,x_n}) - \psi(R_{s}^{L,x_n}) \right) e^{-\delta s}. \]
Further we know (see e.g. [5]) that
\[
\sum_{R_{s-}^{L,x} \neq R_{s-}^{L,x}, \lambda \leq \sigma^*} \left( \psi(R_{s-}^{L,x}) - \psi(R_{s-}^{L,x}) \right) e^{-\delta s} - \int_0^{\sigma^*} e^{-\delta s} \left( \int_0^{R_{s-}^{L,x}} \psi(R_{s-}^{L,x} - y) dF_Y(y) - \psi(R_{s-}^{L,x}) \right) ds
\]
is a martingale. Therefore taking expectations on both sides yields
\[
\mathbb{E} e^{-\delta \sigma^*} \psi(R_{\sigma^*}^{L,x}) + \int_0^{\sigma^*} e^{-\delta s} dL_s \leq \psi(x_n) + \mathbb{E} \left( \int_0^{\sigma^*} e^{-\delta s} \lambda e^{-\delta s} \left( \int_0^{R_{s-}^{L,x}} \psi(R_{s-}^{L,x} - y) dF_Y(y) - \psi(R_{s-}^{L,x}) \right) ds \right).
\]
Because of \( R_{s-}^{L,x} \in B \) for \( s \in [0, \tau^*] \) we have \( \mathcal{L}_\psi(R_{s-}^{L,x}) < -\delta \xi \). We can use this to derive
\[
\mathbb{E} e^{-\delta \sigma^*} V(R_{\sigma^*}^{L,x}) + \int_0^{\sigma^*} e^{-\delta s} dL_s + \mathbb{E} \left( \int_0^{\sigma^*} e^{-\delta s} \xi ds + e^{-\delta \sigma^*} \xi I_{\tau^* = \tau_n} \right) \leq V(x_n) + \gamma_n,
\]
where \( \gamma_n = \psi(x_n) - V(x_n) \) converges to zero. Therefore choose \( n \) large enough such that \( \gamma_n \leq \frac{1}{2} \mathbb{E} \left( \int_0^{\sigma^*} e^{-\delta s} \xi ds + e^{-\delta \sigma^*} \xi I_{\tau^* = \tau_n} \right) \). For arbitrary \( L \) we arrive at
\[
\mathbb{E} e^{-\delta \sigma^*} V(R_{\sigma^*}^{L,x}) + \int_0^{\sigma^*} e^{-\delta s} dL_s + \frac{1}{2} \mathbb{E} \left( \int_0^{\sigma^*} e^{-\delta s} \xi ds + e^{-\delta \sigma^*} \xi I_{\tau^* = \tau_n} \right) \leq V(x_n).
\]
This leads to the following contradiction to the dynamic programming principle:
\[
V(x_n) + \frac{1}{2} \mathbb{E} \left( \int_0^{\sigma^*} e^{-\delta s} \xi ds + e^{-\delta \sigma^*} \xi I_{\tau^* = \tau_n} \right) =
\sup_{L \in \Pi} \mathbb{E} e^{-\delta \sigma^*} V(R_{\sigma^*}^{L,x}) + \int_0^{\sigma^*} e^{-\delta s} dL_s + \frac{1}{2} \mathbb{E} \left( \int_0^{\sigma^*} e^{-\delta s} \xi ds + e^{-\delta \sigma^*} \xi I_{\tau^* = \tau_n} \right) \leq V(x_n).
\]
If there is a positive probability for the event \( \tau^* = 0 \) which is only possible if \( \tau^* = \tau_n \), then for the second term above \( e^{-\delta \tau^*} \xi I_{\tau^* = \tau_n} \) > 0. Therefore \( \mathbb{E} \left( \int_0^{\sigma^*} e^{-\delta s} \xi ds + e^{-\delta \sigma^*} \xi I_{\tau^* = \tau_n} \right) > 0 \) holds and leads indeed to a contradiction.

2.3 Uniqueness

The following comparison principle allows us to decide whether a viscosity supersolution dominates another viscosity subsolution by looking at their initial value. Since every viscosity solution is both a subsolution and supersolution, this will imply uniqueness for a given initial value. Actually in our situation we have to modify the proof presented by Azcuén and Muler [3]. Although quite technical, the arguments are based on an appropriate combination of standard arguments from viscosity theory.

Proposition 2.4. Let for all \( x > 0 \) the functions \( u_1(x) \) and \( u_2(x) \) be a viscosity sub- and supersolution, respectively, that satisfies the conditions fulfilled by the value function (locally Lipschitz, \( u(y) - u(x) \geq y - x \) and some linear growth \( u(x) \leq k_1 x + k_2 \)). If \( u_1(0) \leq u_2(0) \), then \( u_1(x) \leq u_2(x) \) for all \( x \in [0, \infty) \).

Proof. The result will be shown by contradiction. Assume there exists some \( x_0 > 0 \) such that \( u_1(x_0) - u_2(x_0) > 0 \). Let \( \gamma > 0 \) be a constant and define \( \tilde{u}_1(x) = e^{-\gamma x} u_1(x) \) and \( \tilde{u}_2(x) = e^{-\gamma x} u_2(x) \). Because \( u_1 \) and \( u_2 \) fulfill a linear growth condition, these functions are positive and bounded. If we choose \( \gamma \) small enough we get by continuity that \( \tilde{u}_1(x_0) - \tilde{u}_2(x_0) > 0 \). Therefore
\[
0 < \max_{x \geq 0} \left( \tilde{u}_1(x) - \tilde{u}_2(x) \right) = M < \infty,
\]
with a maximizing argument \( x^* \). Further we have
\[
\tilde{u}_1(y) - \tilde{u}_1(x) \leq m, \quad \tilde{u}_2(y) - \tilde{u}_2(x) \leq m,
\]
for all \( x, y \geq x_0 \) with \( y > x \), and
\[
\tilde{u}_1(y) - \tilde{u}_1(x) \leq m, \quad \tilde{u}_2(y) - \tilde{u}_2(x) \leq m,
\]
for some $m > 0$. Define the set $A$ by

$$ A = \{(x, y) \mid 0 \leq x \leq y\}. $$

In the following we need the function

$$ \phi_\nu(x, y) := \bar{u}_1(x) - \bar{u}_2(y) - \frac{\nu}{2}(x - y)^2 - \frac{2m}{\nu^2(y - x) + \nu}, $$

and

$$ M_\nu := \max_{x, y \in A} \phi_\nu(x, y), $$

with the maximizer $(x_\nu, y_\nu)$. We have

$$ M_\nu \geq \phi_\nu(x^*, x^*) = M - \frac{2m}{\nu^*}, $$

which is positive for $\nu$ large enough, leading to

$$ \liminf_{\nu \to \infty} M_\nu \geq M > 0. $$

To ensure differentiability at the points $x_\nu$ and $y_\nu$ one needs to establish that $(x_\nu, y_\nu)$ is not an element of the boundary of $A$ (the proof of which is postponed to Lemma 2.5 after the end of this proof). In the next step we define two test functions, such that we can use that $\bar{u}_1$ and $\bar{u}_2$ are viscosity sub- and supersolutions to a slightly modified problem

$$ \psi(x) = \bar{u}_2(y_\nu) + \frac{\nu}{2}(x - y_\nu)^2 + \frac{2m}{\nu^2(y_\nu - x) + \nu} + \phi_\nu(x_\nu, y_\nu), $$

$$ \varphi(y) = \bar{u}_1(x_\nu) - \frac{\nu}{2}(x_\nu - y)^2 - \frac{2m}{\nu^2(y - x_\nu) + \nu} - \phi_\nu(x_\nu, y_\nu). $$

$\psi$ and $\varphi$ are continuously differentiable functions. Further $\bar{u}_1(x) - \psi(x) = \phi_\nu(x, y_\nu) - \phi_\nu(x_\nu, y_\nu)$ reaches a maximum equal to zero in $x_\nu$. On the other hand $\bar{u}_2(y) - \varphi(y) = -\phi_\nu(x_\nu, y) + \phi_\nu(x_\nu, y_\nu)$ reaches a minimum equal to zero in $y_\nu$. Because $\bar{u}_1$ and $\bar{u}_2$ are viscosity sub- and supersolutions of the original HJB equation, $\bar{u}_1$ and $\bar{u}_2$ are viscosity sub- and supersolutions of the equation

$$ \max \left\{ 1 - e^{\gamma x} (\gamma u(x) + u'(x)), (c + ix)(\gamma u(x) + u'(x)) - (\delta + \lambda)u(x) + \lambda \int_0^x u(x - y) e^{-\gamma y} dF(y) \right\} = 0. $$

In the points $x_\nu$ and $y_\nu$ we get

$$ \max \left\{ 1 - e^{\gamma x} (\gamma \bar{u}_1(x_\nu) + \psi'(x_\nu)), (c + ix_\nu)(\gamma \bar{u}_1(x_\nu) + \psi'(x_\nu)) - (\delta + \lambda)\bar{u}_1(x_\nu) + \lambda \int_0^{x_\nu} \bar{u}_1(x_\nu - y) e^{-\gamma y} dF(y) \right\} \geq 0, $$

$$ \max \left\{ 1 - e^{\gamma y} (\gamma \bar{u}_2(y_\nu) + \varphi'(y_\nu)), (c + iy_\nu)(\gamma \bar{u}_2(y_\nu) + \varphi'(y_\nu)) - (\delta + \lambda)\bar{u}_2(y_\nu) + \lambda \int_0^{y_\nu} \bar{u}_2(y_\nu - y) e^{-\gamma y} dF(y) \right\} \leq 0. $$

In addition we have that $\varphi'(y_\nu) = \psi'(x_\nu) = \nu(x_\nu - y_\nu)$. Notice that $\max\{A, B\} \leq \max\{C, D\}$ implies $A \leq C \lor B \leq D$. We start with looking at $B \leq D$,

$$ (c + iy_\nu)(\gamma \bar{u}_2(y_\nu) + \nu(x_\nu - y_\nu)) - (c + ix_\nu)(\gamma \bar{u}_1(x_\nu) + \nu(x_\nu - y_\nu)) + (\delta + \lambda)(\bar{u}_1(x_\nu) - \bar{u}_2(y_\nu)) $$

$$ \leq \lambda \left( \int_0^{x_\nu} \bar{u}_1(x_\nu - y) e^{-\gamma y} dF(y) - \int_0^{y_\nu} \bar{u}_2(y_\nu - y) e^{-\gamma y} dF(y) \right). \quad (11) $$

From

$$ \phi_\nu(x_\nu, x_\nu) + \phi_\nu(y_\nu, y_\nu) \leq 2\phi_\nu(x_\nu, y_\nu) $$

we immediately get

$$ \bar{u}_1(x_\nu) - \bar{u}_2(x_\nu) + \bar{u}_1(y_\nu) - \bar{u}_2(y_\nu) - \frac{4m}{\nu} \leq 2 \left( \bar{u}_1(x_\nu) - \bar{u}_2(y_\nu) - \frac{\nu}{2}(x_\nu - y_\nu)^2 - \frac{2m}{\nu^2(y_\nu - x_\nu) + \nu} \right). $$
This yields, together with (10),
\[ \nu(x_\nu - y_\nu)^2 \leq \tilde{u}_1(x_\nu) - \tilde{u}_1(y_\nu) + \tilde{u}_2(x_\nu) - \tilde{u}_2(y_\nu) + 4m \frac{y_\nu - x_\nu}{\nu(y_\nu - x_\nu) + 1} \leq 6m |y_\nu - x_\nu| \]
and in particular
\[ |y_\nu - x_\nu| \leq \frac{6m}{\nu}. \tag{12} \]
Now let \((\nu_n)_{n \in \mathbb{N}}\) be such that \((x_\nu, y_\nu)\) converges to \((\bar{x}, \bar{y})\) as \(\nu_n \to \infty\). From (12) we get that \(\bar{x} = \bar{y}\).
Using (11) we get
\[ (c + \nu_1)\gamma(\tilde{u}_2(\bar{x}) - \tilde{u}_1(\bar{x})) + (\delta + \lambda)(\tilde{u}_1(\bar{x}) - \tilde{u}_2(\bar{x})) \leq \lambda \left( \int_0^\gamma e^{-\gamma y}(\tilde{u}_1(\bar{x} - y) - \tilde{u}_2(\bar{x} - y))dF_\gamma(y) \right). \tag{13} \]
The right-hand side of (13) is smaller than \(\lambda M\). If we choose \(\gamma\) small enough we derive
\[ M \leq \liminf_{\nu \to \infty} M_\nu \leq \lim_{n \to \infty} M_{\nu_n} = \tilde{u}_1(\bar{x}) - \tilde{u}_2(\bar{x}) \leq \frac{\lambda}{\delta + \lambda} M, \]
which is a contradiction.
Now we concentrate on \(A \leq C\) and observe that
\[ e^{\gamma x_\nu}(\gamma \tilde{u}_1(x_\nu) + \psi'(x_\nu)) \leq e^{\gamma y_\nu}(\gamma \tilde{u}_2(y_\nu) + \varphi'(y_\nu)). \]
This implies
\[ \nu(x_\nu - y_\nu)(e^{\gamma x_\nu} - e^{\gamma y_\nu}) \leq (e^{\gamma y_\nu} \tilde{u}_2(y_\nu) - e^{\gamma x_\nu} \tilde{u}_1(x_\nu)). \]
For \(\gamma\) small enough we have \(e^{\gamma y_\nu} \tilde{u}_2(y_\nu) - e^{\gamma x_\nu} \tilde{u}_1(x_\nu) \approx \tilde{u}_2(y_\nu) - \tilde{u}_1(x_\nu)\) so that
\[ 0 < M \leq M_\nu = \phi_\nu(x_\nu, y_\nu) \leq \tilde{u}_1(x_\nu) - \tilde{u}_2(y_\nu) \leq (y_\nu - x_\nu) \frac{\nu}{\gamma}(e^{\gamma x_\nu} - e^{\gamma y_\nu}) \leq 0. \]

From the comments above, Proposition 2.4 implies the uniqueness of the viscosity solution for a given initial condition \(v(0) = v_0\).

**Lemma 2.5.** \((x_\nu, y_\nu)\) is not an element of the boundary of \(A\).

**Proof.** First look at
\[ \phi_\nu(0, 0) = \tilde{u}_1(0) - \tilde{u}_2(0) - \frac{2m}{\nu} < 0, \]
\[ \lim_{b \to \infty} \phi_\nu(x, b) = \tilde{u}_1(x) - \tilde{u}_2(b) - \frac{\nu}{2} (x - b)^2 - \frac{2m}{\nu^2 (b - x) + \nu} = -\infty. \]
The next step is to examine the left-hand derivative at the boundary of \(A\) along the diagonal. For all \(x > 0\),
\[ \limsup_{h \to 0^+} \frac{\phi_\nu(x, x - h)}{h} = \limsup_{h \to 0^+} \frac{1}{h} \left( \tilde{u}_1(x) - \tilde{u}_1(x - h) - \frac{2m}{\nu} + \frac{\nu}{2} h^2 + \frac{2m}{\nu^2 h + \nu} \right) \leq \limsup_{h \to 0^+} \left( m + \frac{\nu}{2} h - \frac{2m}{\nu h + 1} \right) = -m < 0. \]
By continuity it follows from \(\phi_\nu(0, 0) < 0\) that \(\phi_\nu(y, 0) < 0\) for \(y \in [0, \rho]\) and some \(\rho > 0\). Now for \(y > \rho\) we observe
\[ \limsup_{h \to 0^+} \frac{\phi_\nu(y, y - h)}{h} = \limsup_{h \to 0^+} \frac{1}{h} \left( \tilde{u}_1(y) - \tilde{u}_1(h) + \frac{\nu}{2} h^2 - \nu y + \frac{2m}{(\nu - h + 1)(\nu y + 1)} \right) \leq \limsup_{h \to 0^+} \left( \gamma \tilde{u}_1(0) - e^{-\gamma h + o(h)} + \frac{\nu}{2} h - \nu y + \frac{2m}{(\nu - h + 1)(\nu y + 1)} \right) = \gamma \tilde{u}_1(0) - 1 - \nu y + \frac{2m}{(\nu + 1)^2}. \tag{14} \]
which is negative for \( \nu \) large enough. Here the inequality in (14) holds because the lower and upper linear growth conditions imply \(-h \geq u_1(0) - u_1(h)\) and consequently
\[-h \geq \bar{u}_1(0) - e^{\gamma h} \bar{u}_1(h) \Rightarrow -he^{-\gamma h} \geq \bar{u}_1(0) - \bar{u}_1(h) + (e^{-\gamma h} - 1) \bar{u}_1(0) \Rightarrow \gamma \bar{u}_1(0)h - he^{-\gamma h} + o(h^2) \geq \bar{u}_1(0) - \bar{u}_1(h).\]

Hence we have proved that \((x_n, y_n)\) does not belong to the boundary of \(A\) (negative value in \((0, 0)\) and in every direction towards the boundary of \(A\) negative derivatives and a negative limit for the argument \((x, b)\) if \(b \to \infty\).

\[\square\]

2.4 Characterization of the value function

In contrast to some optimization problems in a diffusion framework the dividend maximization problem in our setup lacks an initial condition. In Proposition 2.6 we will prove that every viscosity supersolution to (5) which fulfills a linear growth condition dominates the value function. This together with Proposition 2.4 allows us to define

\[V(0) = \inf \{u(0) \mid u \text{ is a viscosity solution to the HJB equation and fulfills a linear growth condition}\}.\]

Because of the comparison principle any other choice of an initial value will lead to a contradiction to Proposition 2.6, since for any suitable viscosity solution \(u\) with \(u(0) < V(0)\) we would have \(u(x) < V(x)\) for at least \(x \in [0, \epsilon)\) and \(u(x) \leq V(x)\) for all \(x > 0\).

For a viscosity supersolution \(u_1\) we have almost everywhere

\[u_1'(x) \leq \frac{1}{c + ix} \left( (\delta + \lambda) u_1(x) - \lambda \int_0^x u_1(x-y)dF_Y(y) \right) \leq \frac{\delta + \lambda}{c + \lambda} u_1(x).\]

Throughout this section we need a sequence of non-negative functions \(\{v_n(x)\}_{n \in \mathbb{N}}\) with the following properties:

- \(v_n\) is continuously differentiable with
  \[1 \leq v_n' \leq \frac{\delta + \lambda}{c + \lambda} v_n\]

- \(v_n(x) \leq k_1 x + k_2\) for some positive constants \(k_1, k_2\)

- \(v_n\) converges uniformly to the absolutely continuous supersolution \(u_1\) of (5) on compact sets and \(v_n'\) converges to \(u_1'\) almost everywhere. Further \(v_n(x) = 0\) for \(x < 0\).

Such a sequence exists due to [15] and [3].

**Proposition 2.6.** An absolutely continuous supersolution \(u_1\) of the HJB equation (5) fulfilling a linear growth condition dominates the value function, \(u_1(x) \geq V(x)\).

**Proof.** Let \(L = (L_t)_{t \geq 0}\) be an admissible strategy. The controlled process is \(R^L = (R^L_t)_{t \geq 0}\) and \(R^L_t = x\) with ruin time \(\tau\). Let \(v_n(x)\) be a continuously differentiable element from the sequence defined above. We have

\[v_n(R^L_{(t\wedge \tau)})e^{-\delta(t\wedge \tau)} = \frac{v_n(x) + \int_0^{(t\wedge \tau)} v_n'(R^L_s) e^{-\delta s} dR^L_s}{\int_0^{(t\wedge \tau)} v_n'(R^L_s) e^{-\delta s} ds} - \frac{\delta}{\int_0^{(t\wedge \tau)} v_n'(R^L_s) e^{-\delta s} ds}.\]

having in mind that claim occurrences lead to \(L_{x-} \neq L_x\) and singular dividend payments (hump sums) lead to \(L_{x+} \neq L_x\), we get from the construction of the reserve process

\[\int_0^{(t\wedge \tau)} v_n'(R^L_s) e^{-\delta s} dR^L_s = \int_0^{(t\wedge \tau)} e^{-\delta s} \left( c + i R^L_s \right) v_n'(R^L_s) ds - \int_0^{(t\wedge \tau)} v_n'(R^L_s) dL^c_s + \sum_{R^L_{s-} \neq R^L_s, s < (t\wedge \tau)} \left( v_n(R^L_{s-}) - v_n(R^L_s) \right) e^{-\delta s} + \sum_{R^L_{s+} \neq R^L_s, s < (t\wedge \tau)} \left( v_n(R^L_{s+}) - v_n(R^L_s) \right) e^{-\delta s}.\]
Using the continuity of $v'_n$ and $R^L_{s+} - R^L_s = -(L_{s+} - L_s)$ we can write

$$v_n(R^L_{s+}) - v_n(R^L_s) = - \int_0^{L_{s+} - L_s} v'_n(R_s - \gamma) \, d\gamma.$$  

Further we use the martingale $(M_t)_{t \geq 0}$

$$M_t = \sum_{R^L_s \neq R^L_t, \, s < t} (v_n(R^L_s) - v_n(R^L_{s-}) \, e^{-\delta s} - \lambda \int_0^t e^{-\delta s} \left( \int_0^{R^L_{s-}} v_n(R^L_s - y) \, dF_Y(y) - v_n(R^L_{s-}) \right) \, ds,$$

which is the compensated process, see [5]. We arrive at

$$v_n(R^L_{t\wedge \tau}) e^{-\delta (t \wedge \tau)} = v_n(x) + \int_0^{(t \wedge \tau)} e^{-\delta s} \left[ (c + iR_s) v'_n(R^L_s) - (\delta + \lambda) v_n(R^L_s) + \lambda \int_0^{R^L_s} v_n(R^L_s - y) \, dF_Y(y) \right] \, ds$$

$$- \int_0^{(t \wedge \tau)} v'_n(R^L_s) \, dL^L_s - \sum_{L_{s+} \neq L_s, \, s < (t \wedge \tau)} e^{-\delta s} \int_0^{L_{s+} - L_s} v'_n(R^L_s - \gamma) \, d\gamma + M_{(t \wedge \tau)}.$$  

Now we use $v'_n \geq 1$ and can estimate

$$- \int_0^{(t \wedge \tau)} v'_n(R^L_s) \, dL^L_s - \sum_{L_{s+} \neq L_s, \, s < (t \wedge \tau)} e^{-\delta s} \int_0^{L_{s+} - L_s} v'_n(R^L_s - \gamma) \, d\gamma$$

$$\leq - \int_0^{(t \wedge \tau)} dL^L_s - \sum_{L_{s+} \neq L_s, \, s < (t \wedge \tau)} e^{-\delta s} \int_0^{L_{s+} - L_s} d\gamma = - \int_0^{(t \wedge \tau)} e^{-\delta s} dL_s,$$

which leads to

$$v_n(R^L_{(t \wedge \tau)}) e^{-\delta (t \wedge \tau)} \leq v_n(x) + \int_0^{(t \wedge \tau)} e^{-\delta s} L_n(R^L_s) \, ds - \int_0^{(t \wedge \tau)} e^{-\delta s} dL_s + M_{(t \wedge \tau)}.$$  

The next steps are taking expectations, examining the validity of taking the limit $t \to \infty$ and letting $n \to \infty$. This will give the desired result.

Starting with

$$\mathbb{E} \left( v_n(R^L_{(t \wedge \tau)}) e^{-\delta (t \wedge \tau)} \right) \leq v_n(x) + \mathbb{E} \left( \int_0^{(t \wedge \tau)} e^{-\delta s} L_n(R^L_s) \, ds \right) - \mathbb{E} \left( \int_0^{(t \wedge \tau)} e^{-\delta s} dL_s \right),$$  \hspace{1cm} (16)

we have to find integrable bounds for every summand to justify the interchange of limit and integration. Because $L_s$ is increasing, we get by monotone convergence

$$\lim_{t \to \infty} \mathbb{E} \left( \int_0^{(t \wedge \tau)} e^{-\delta s} dL_s \right) = \mathbb{E} \left( \int_0^{T} e^{-\delta s} dL_s \right) = V_L(x).$$

Next we look at the second summand on the right hand side, use the estimates for the first derivative (15), the linear growth and the reserve from above to get the integrable upper bound

$$(c + ix) v'_n(x) - (\lambda + \delta) v_n(x) + \lambda \int_0^y v_n(x - y) \, dF_Y(y) \leq \lambda \int_0^y v_n(x - y) \, dF_Y(y) \leq \lambda v_n(x),$$

which gives

$$\int_0^{(t \wedge \tau)} e^{-\delta s} L_n(R^L_s) \, ds \leq \int_0^{(t \wedge \tau)} e^{-\delta s} \lambda v_n(R^L_s) \, ds \leq \int_0^{(t \wedge \tau)} e^{-\delta s} \lambda \left( k_1 e^{is} \left( x + c \int_0^\tau e^{-ih} \, dh \right) + k_2 \right) \, ds$$

$$\leq \int_0^\infty e^{-\delta s} \lambda \left( k_1 e^{is} \left( x + c \int_0^\tau e^{-ih} \, dh \right) + k_2 \right) \, ds < \infty$$

11
(recall that we have $i < \delta$), so that by dominated convergence
\[
\lim_{t \to t^+} \mathbb{E} \left( \int_{t}^{t^+} e^{-\delta s} \mathcal{L}_{v_n}(R^L_s) \, ds \right) = \mathbb{E} \left( \int_{0}^{\tau} e^{-\delta s} \mathcal{L}_{v_n}(R^L_s) \, ds \right).
\]

The left hand side of (16) converges to zero by
\[
- \mathbb{E} \left( v_n(R^L_{t^+};(t,t^+))e^{-\delta(t^+)} \right) - \mathbb{E} \left( v_n(R^L_{t};(t,t^+))e^{-\delta(t^+)} I_{(t,t^+)} \right) \leq \mathbb{E} \left( v_n(R^L_{t};(t,t^+))e^{-\delta(t^+)} \right) \leq \mathbb{E} \left( e^{-\delta t} \left( k_1 e^{i \pi} (x + c \int_{0}^{t} e^{-ib} \, dh) + k_2 \right) \right) \to 0.
\]

(recall $v_n(x) = 0$ for $x < 0$). Further $v'_n \to u'_1$ almost everywhere (at points where $u_1$ is differentiable) and $\lim_{n \to \infty} \mathcal{L}_{v_n}(x) = \mathcal{L}_{u_1}(x)$ holds.

We need again an integral upper bound for $\mathcal{L}_{v_n}(R^L_{t}) - \mathcal{L}_{u_1}(R^L_{t})$. This can be obtained from (15) and the linear growth conditions on $v_n$ and $u_1$:

\[
\mathbb{E} \left( v_n(R^L_{t}) \, ds \right) - \mathbb{E} \left( u_1(R^L_{t}) \, ds \right) \leq \left( (c + iR^L_k)u'_n(R^L_k) + (\delta + \lambda) v_n(R^L_k) + \lambda \int_{0}^{R^L_k} v_n(R^L_k - y) \, dF_{Y}(y) \right)
+ (c + iR^L_k)u'_1(R^L_k) + (\delta + \lambda) u_1(R^L_k) + \lambda \int_{0}^{R^L_k} u_1(R^L_k - y) \, dF_{Y}(y)
\leq 2(\delta + \lambda) v_n(R^L_k) + \lambda \int_{0}^{R^L_k} v_n(R^L_k - y) \, dF_{Y}(y) + 2(\delta + \lambda) u_1(R^L_k) + \lambda \int_{0}^{R^L_k} u_1(R^L_k - y) \, dF_{Y}(y)
\leq K \left( u_1(R^L_k) + v_n(R^L_k) \right) e^{-\delta s} \leq K \left( k_1 e^{i \pi} (x + c \int_{0}^{s} e^{-ib} \, dh) + k_2 \right) e^{-\delta s}.
\]

Altogether we arrive at
\[
\lim_{n \to \infty} \mathbb{E} \left( \int_{0}^{\tau} e^{-\delta s} \mathcal{L}_{v_n}(R^L_s) \, ds \right) = \mathbb{E} \left( \int_{0}^{\tau} e^{-\delta s} \mathcal{L}_{u_1}(R^L_s) \, ds \right) \leq 0.
\]

Finally we arrive at
\[
V_L(x) \leq u_1(x) + \mathbb{E} \left( \int_{0}^{\tau} e^{-\delta s} \mathcal{L}_{u_1}(R^L_s) \, ds \right) \leq u_1(x),
\]

which holds for every admissible strategy $L$ resulting in $V(x) \leq u_1(x)$.

The next proposition follows immediately.

**Proposition 2.7.** An admissible strategy $L$ with associated return function $V_L$ which is an absolutely continuous supersolution of the HJB equation fulfills $V = V_L$. Consequently, $L$ is an optimal dividend strategy.

Now we state several auxiliary results which characterize the value function at points of potentially problematic differentiability behaviour. The proofs are in the spirit of Azcue and Muler in [3].

If it is optimal to pay out an amount $a$ immediately, then $V(x) = a + V(x - a)$ so that $V'(x^-) = 1$. If it is optimal to keep the surplus at a level $x$ until the next claim occurrence at time $\tau_1$ and pay out everything exceeding this level we have
\[
V(x) = \mathbb{E} \left( \int_{0}^{\tau_1} (c + ix \, e^{-\delta s}) \, ds + e^{-\delta \tau_1} V(x - Y_1) \right) = \frac{1}{\delta + \lambda} \left( c + ix + \lambda \int_{0}^{x} V(x - y) \, dF_{Y}(y) \right).
\]

The following assertions are needed to prove certain properties of the optimal strategy.

For some $z < 0$, the set $\Pi_z$ will denote the set of admissible strategies $L \in \Pi$ for which the controlled reserve stays below $z$, i.e. $R^L_t \leq z$ for $L \in \Pi_z$ and $t \geq 0$.

Define the operator
\[
\Lambda(x) = c + ix - (\delta + \lambda)V(x) + \lambda \int_{0}^{x} V(x - y) \, dF_{Y}(y).
\]

12
Lemma 2.8. If there is an $\tau > 0$ such that $\Lambda(\tau) = 0$, then $V(x) = \sup_{L \in \Pi_n} V_L(x)$ for $x \in [0, \tau)$.

Proof. The proof is done by induction. Let $\Pi_{(n)}$ be the set of admissible strategies such that for initial reserve $x < \tau$ the claim process stays below $\tau$ till the occurrence of the $n$th claim. The idea of the proof is to construct an $\epsilon$-optimal strategy $L_n \in \Pi_{(n)}$ from a certain $\epsilon/2$-optimal strategy $L_{n-1} \in \Pi_{(n-1)}$ for some $n$ large enough. Because of discounting and $\delta > \epsilon$ we get that $|V_{L_n}(x) - V_L(x)|$ will be small enough to derive the desired result.

First we want to show

$$V(x) = \sup_{L \in \Pi_{(n)}} V_L(x) \quad (17)$$

for all $n \geq 0$. This will be done by induction. Clearly $\Pi_{(0)} = \Pi$ and we have that $V(x) = \sup_{L \in \Pi_{(0)}} V_L(x)$. Let $n > 1$, $\epsilon > 0$ and (17) be fulfilled for $n - 1$. By the induction hypothesis, $L_{n-1} \in \Pi_{(n-1)}$ such that $V(x) - V_{L_{n-1}}(x) < \frac{\epsilon}{2}$. Now we look for a strategy $L_n \in \Pi_{(n)}$ such that $0 \leq V_{L_{n-1}}(x) - V_{L_n}(x) \leq \frac{\epsilon}{2}$. In view of $\Lambda(\tau) = 0$, $L_n$ is defined as follows. Starting at $x < \tau$ apply $L_{n-1}$ as long as the reserve stays below $\tau$. When reaching $\bar{\tau}$ pay out $c + \epsilon \bar{\tau}$ until a claim occurs and use again $L_{n-1}$ with initial capital $\tau - \bar{\tau}$, where $\bar{\tau}$ denotes the random claim size.

As first step we show $V_{L_n}(\tau) \geq V_{L_{n-1}}(\tau) - \frac{\epsilon}{2}$. The initial capital is $R_0^{L_n} = \tau$, $Y_1$, $\tau_1$ denote amount and occurrence time of the first claim. For $0 \leq t < \tau_1$ we have $R_t^{L_n} = \tau$, $L_{n,t} = (c + \bar{\tau})t$ and $R_t^{L_n} = \tau - Y_1$. We get

$$V_{L_n}(\tau) = \mathbb{E}\left( \int_0^{\tau_1} e^{-\delta s} (c + \bar{\tau}) \, ds + e^{-\delta \tau_1} V_{L_{n-1}}(\tau - Y_1) \right)$$

$$= \frac{1}{\delta + \lambda} \left( c + \bar{\tau} + \lambda \int_0^{\tau} V_{L_{n-1}}(\tau - y) \, dF_Y(y) \right)$$

$$\geq \frac{1}{\delta + \lambda} \left( c + \bar{\tau} + \lambda \int_0^{\tau} (V(\tau - y) - \frac{\epsilon}{2}) \, dF_Y(y) \right)$$

$$= \frac{1}{\delta + \lambda} \left( \Lambda(\tau) + (\delta + \lambda)V(\tau)\frac{\epsilon}{2} F(\tau) \right)$$

$$\geq V(\tau) - \frac{\epsilon}{2}.$$

From the following two inequalities we get the required result.

$$V(\tau) \geq V_{L_{n-1}}(\tau) \geq V(\tau) - \frac{\epsilon}{2},$$

$$V(\tau) \geq V_{L_n}(\tau) \geq V(\tau) - \frac{\epsilon}{2},$$

which gives

$$V_{L_n}(\tau) - V_{L_{n-1}}(\tau) \geq V(\tau) - \frac{\epsilon}{2} - V(\tau) = -\frac{\epsilon}{2}.$$

Now we deal with the case $0 \leq x < \tau$. We have to distinguish between paths of the process controlled by $L_n$ which reach $\tau$ in finite time (the set of these paths is denoted by $\mathcal{P}_1$) and those which do not. Let $\bar{\tau}$ be the first time a path from $\mathcal{P}_1$ reaches $\tau$. We can split the value of the strategy $L_n$ as follows

$$V_{L_n}(x) = \mathbb{E} \left( I_{\mathcal{P}_1} \int_0^{\tau_n} e^{-\delta s} dL_{n,s} \right) + \mathbb{E} \left( I_{\bar{\tau}} \int_0^{\tau_n} e^{-\delta s} dL_{n,s} \right)$$

$$= \mathbb{E} \left( I_{\mathcal{P}_1} \int_0^{\tau} e^{-\delta s} dL_{n,s} \right) + \mathbb{E} \left( e^{-\delta \tau} V_{L_n}(\tau) + \mathbb{E} \left( I_{\bar{\tau}} \int_0^{\tau_n} e^{-\delta s} dL_{n,s} \right) \right).$$

Because of the definition of the strategy $L_n$ we have that in $\mathcal{P}_1^c$ and in $\mathcal{P}_1$ for $t < \tau$ the paths $R_t^{L_n}$ and $R_t^{L_n-1}$ are identical. Therefore we arrive at

$$V_{L_{n-1}}(x) - V_{L_n}(x) = \mathbb{E} \left( I_{\mathcal{P}_1} e^{-\delta \tau} (V_{L_{n-1}}(\tau) - V_{L_n}(\tau)) \right)$$

$$\leq \mathbb{E} \left( I_{\mathcal{P}_1} e^{-\delta \tau} \frac{\epsilon}{2} \right) \leq \frac{\epsilon}{2}.$$
In the end we have to show that for every \( \epsilon > 0 \) there exists a strategy \( \hat{L} \in \Pi_{\pi} \) such that \( V(x) - V_L(x) < \epsilon \) for \( x \in [0, \overline{\tau}] \). First define \( t_1 \) such that

\[
e^{-\delta t_1} < \frac{\epsilon}{8V(\overline{\tau})},
\]

and \( n \geq 1 \) large enough such that

\[
P(N_{t_1} \geq n) = \sum_{k \geq n} \frac{e^{-\lambda t_1} (\lambda t_1)^k}{k!} \leq \frac{\epsilon}{8V(\overline{\tau})}.
\]

Let \( L_n \in \Pi_{(n)} \) be an \( \epsilon/2 \)-optimal strategy for all \( x \in [0, \overline{\tau}] \). Let \( \hat{\tau} \) the first time a path of \( (R_t^L)_{t \geq 0} \) exceeds \( \overline{\tau} \). The set \( \mathcal{P}_2 \) consists of all paths such that \( \hat{\tau} < \infty \). For \( t < \hat{\tau} \) we define \( \hat{L} = L_n \), if \( t = \hat{\tau} \) the strategy \( \hat{L} \) pays out immediately \( \overline{\tau} \) and the incoming premiums till the next claim occurrence which leads to ruin. As before the value of the strategy \( L_n \) as well as for \( \hat{L} \) can be written in the following form,

\[
V_{L_n}(x) = \mathbb{E}\left( \int_0^{L_n} e^{-\delta s} dL_{n,s} \right) + \mathbb{E}\left( \int_0^{t_{n,s}} e^{-\delta s} dL_{n,s} \right)
\]

\[
= \mathbb{E}\left( \int_0^{t_{n,s}} e^{-\delta s} dL_{n,s} \right) + \mathbb{E}\left( \int_0^{\hat{\tau}} e^{-\delta s} dL_{n,s} \right) + \mathbb{E}(I_{(\hat{\tau} < \infty)} V_{\hat{L}}(\overline{\tau}))
\]

\[
\leq \mathbb{E}\left( \int_0^{t_{n,s}} e^{-\delta s} dL_{n,s} \right) + \mathbb{E}\left( \int_0^{\hat{\tau}} e^{-\delta s} dL_{n,s} \right) + \mathbb{E}(I_{(\hat{\tau} < \infty)} V(\overline{\tau}))
\]

Since \( R_t^{L_n} \) and \( R_t^L \) are identical on \( \mathcal{P}_2 \) and for \( t < \hat{\tau} \) we get

\[
|V_{L_n}(x) - V_L(x)| \leq 2 \mathbb{E}(I_{(\hat{\tau} < \infty)} V(\overline{\tau})),
\]

Because \( L_n \in \Pi_{(n)} \) we have \( \{\hat{\tau} < t_1\} \subset \{N_{t_1} \geq n\} \), furthermore we have \( \mathcal{P}_2 = \{\hat{\tau} < \infty\} \subset \{\hat{\tau} \geq t_1\} \cup \{N_{t_1} \geq n\} \). We get

\[
\mathbb{E}(I_{(\hat{\tau} < t_1)} e^{-\delta \hat{\tau}}) \leq \mathbb{E}(I_{(\hat{\tau} < \infty)} e^{-\delta \hat{\tau}}) \leq e^{-\delta t_1} + P(\{N_{t_1} \geq n\}) < \frac{\epsilon}{4V(\overline{\tau})},
\]

which gives

\[
|V_{L_n}(x) - V_L(x)| < \frac{\epsilon}{2}.
\]

The required result follows from

\[
V(x) - V_L(x) \leq V(x) - V_{L_n}(x) + |V_{L_n}(x) - V_L(x)| < \epsilon.
\]

\[\Box\]

**Lemma 2.9.** If there is an \( \overline{\tau} > 0 \) such that \( V(\overline{\tau}) = 1 \), then \( V(x) = \sup_{L \in \Pi_{\pi}} V_L(x) \) for all \( x \in [0, \overline{\tau}] \).

**Proof.** We have to show that for every \( \epsilon > 0 \) we are able to find a strategy \( \hat{L} \in \Pi_{\pi} \) such that \( 0 \leq V(x) - V_{\hat{L}}(x) < \epsilon \) for all \( x \in [0, \overline{\tau}] \). Let

\[
D = \frac{c + i\overline{x}}{\delta} \ln \left( \frac{2V(\overline{\tau})}{\epsilon} \right)
\]

and define a sequence \( \{x_n\}_{n \in \mathbb{N}} \) with

\[
x_n = \overline{\tau} - \frac{D}{n}.
\]

14
Further we need a sequence \( \{ h_n \}_{n \in \mathbb{N}} \) defined by

\[
h_n = \frac{V(x_n) - V(\pi)}{x_n - \pi} - 1.
\]

Because of \( V'(\pi) = 1 \) we have that \( h_n \to 0 \) for \( n \to \infty \). Choose \( n_0 \) such that \( h_{n_0} < \frac{\epsilon}{2m} \). A further specification of the size of \( n_0 \) will be needed in the end of the proof.

The proof needs two steps: first one fixes a sequence of strategies such that on a certain level we get an \( \frac{2}{3} \)-optimal strategy and the differences of the values of these strategies form a decreasing sequence. In a second step these ingredients are used to define an \( \epsilon \)-optimal strategy within the set \( \Pi_n \).

**Step 1:**
Take a strategy \( L \in \Pi \) such that \( V(x) - V_L(x) < \frac{\epsilon}{8m} \). Now define in a recursive way the following set of strategies \( \{ L_n \}_{n \geq 0} \). For \( n = 0 \) set \( L_0 = L \). For \( n > 0 \) and initial capital \( x \leq x_{n_0} \) follow the strategy \( L \) as long as \( R_L^t < \pi \) and as \( R_L^t \) reaches \( \pi, \) pay out immediately the difference \( \pi - x_{n_0} \) and follow \( L_{n-1} \) with initial capital \( x_{n_0} \). If \( x \in [x_{n_0}, \pi] \), pay out \( x - x_{n_0} \) and follow \( L_{n-1} \).

The idea behind this procedure is to find an estimate for the time the process stays below \( \pi \) before crossing \( \pi \). Under the strategy \( L_n \), the interval \( [x_{n_0}, \pi] \) has to be passed more than \( n \) times.

The first thing to show is \( V(x) - V_{L_{n_0}} < \frac{\epsilon}{2} \) for all \( x \in [0, \pi] \).

We start with showing that \( V(x) - V_{L_1}(x) < \frac{\epsilon}{2m} \) for all \( x \in [0, \pi] \).

For \( x = \pi \) we have

\[
V(\pi) - V_{L_1}(\pi) \leq V(x_{n_0}) + (1 + h_{n_0})(\pi - x_{n_0}) - (\pi - x_{n_0}) + V_L(x_{n_0})
\]

\[
= V(x_{n_0}) - V_L(x_{n_0}) + h_{n_0}(\pi - x_{n_0}) \leq \frac{\epsilon}{4m_0},
\]

because of \( \pi - x_{n_0} = \frac{m_0}{x_{n_0}}, h_{n_0} \leq \frac{m_0}{x_{n_0}}, (1 + h_{n_0})(\pi - x_{n_0}) = V(\pi) - V(x_{n_0}) \) and \( V(\pi) \geq \pi - x + V(x) \).

If \( x \in [x_{n_0}, \pi] \) we get with \( V(\pi) = (1 + h_{n_0})(\pi - x_{n_0}) + V(x_{n_0}) \),

\[
V(x) - V_{L_1}(x) \leq V(\pi - (\pi - x) - (x - x_{n_0} + V_L(x_{n_0}))
\]

\[
= V(x_{n_0}) + h_{n_0}(\pi - x_{n_0}) + \pi - x - x_{n_0} + V_L(x_{n_0})
\]

\[
= V(x_{n_0}) - V_L(x_{n_0}) + h_{n_0}(\pi - x_{n_0}) \leq \frac{\epsilon}{4m_0},
\]

with the same arguments as above.

In the end we look at \( x \in [0, x_{n_0}] \). Let \( \mathcal{P}_3 \) be the set of paths of \( R^{L^1} \) with initial capital \( x \) such that \( x_{n_0} \) is reached in finite time, let \( \tau_x \) be the first time such that this is done by a path from \( \mathcal{P}_3 \). We derive

\[
V_{L_1}(x) = \mathbb{E} \left( I_{\mathcal{P}_3} \int_0^{\tau_x} e^{-\delta s} dL_{1,s} \right) + \mathbb{E} \left( I_{\mathcal{P}_3} \int_0^{\tau_x} e^{-\delta s} dL_{1,s} \right)
\]

Because the paths of \( R^L \) and \( R^{L^1} \) coincide in \( \mathcal{P}_3 \), and in \( \mathcal{P}_3 \) for \( t < \tau_x \) we get

\[
| V_{L_1}(x) - V_L(x) | = \mathbb{E} \left( I_{\mathcal{P}_3} e^{-\delta \tau_x} \right) | V_{L_1}(x_{n_0}) - V_L(x_{n_0}) |.
\]

This together with the above estimates, \( \mathbb{E} (I_{\mathcal{P}_3} e^{-\delta \tau_x}) \leq 1 \), yields

\[
| V(x) - V_{L_1}(x) | \leq | V(x) - V_L(x) | + | V_L(x) - V_{L_1}(x) |
\]

\[
\leq | V(x) - V_L(x) | + | V_L(x_{n_0}) - V_{L_1}(x_{n_0}) |
\]

\[
\leq | V(x) - V_L(x) | + | V(x_{n_0}) - V_{L}(x_{n_0}) | + | V(x_{n_0}) - V_{L_1}(x_{n_0}) | \leq \frac{\epsilon}{2m_0}.
\]

Now we want for \( n \geq 2 \) and \( x \in [0, \pi] \) that \( | V_{L_n}(x) - V_{L_{n-1}}(x) | \leq | V_{L_{n-1}}(x_{n_0}) - V_{L_{n-2}}(x_{n_0}) | \) holds.

For \( x \in [x_{n_0}, \pi] \) and \( n \geq 1 \) we get the result immediately from \( V_{L_n}(x) = x - x_{n_0} + V_{L_{n-1}}(x_{n_0}) \).
Let \( x \in [0, x_{n_0}) \) and denote by \( \mathcal{P}_4 \) the set of paths of \( R^L \) such that \( \tau \) is reached in finite time, \( \tau_x \) denoting the first time of such an event. We obtain
\[
V_{L_n}(x) = \mathbb{E} \left( I_{\mathcal{P}_4} \int_0^{t_{L_n}} e^{-\delta s} \, dL_{n,s} \right) + \mathbb{E} \left( I_{\mathcal{P}_4^c} \int_0^{t_{L_n}} e^{-\delta s} \, dL_{n,s} \right).
\]

As before the paths of \( R^{L_n} \) and \( R^{L_n-1} \) coincide on \( \mathcal{P}_4^c \) and on \( \mathcal{P}_4 \) for \( t < \overline{\tau}_x \). Therefore
\[
|V_{L_n}(x) - V_{L_{n-1}}(x)| = \mathbb{E} (e^{-\delta \tau}) |V_{L_{n-1}}(x_{n_0}) - V_{L_{n-2}}(x_{n_0})| \leq |V_{L_{n-1}}(x_{n_0}) - V_{L_{n-2}}(x_{n_0})|.
\]
We arrive at
\[
V(x) - V_{L_{n_0}}(x) = V(x) - V_{L_1}(x) + \sum_{n=2}^{n_0} (V_{L_{n-1}}(x) - V_{L_{n}}(x))
\]
\[
\leq V(x) - V_{L_1}(x) + (n_0 - 1)|V_{L_1}(x_{n_0}) - V_{L_1}(x_{n_0})|
\]
\[
\leq \frac{e}{2n_0} + (n_0 - 1)(|V_{L_1}(x_{n_0}) - V(x_{n_0})| + |V(x_{n_0}) - V_{L_1}(x_{n_0})|)
\]
\[
\leq \frac{e}{2n_0} + (n_0 - 1)(\frac{e}{4n_0} + \frac{e}{8n_0}) \leq \frac{e}{2}.
\]

Step 2:
Now we identify a strategy \( L \in \Pi_{\Phi} \) such that \( V_{L_{n_0}}(x) - V_L(x) < \frac{e}{2} \) for all \( x \in [0, \overline{\tau}] \). In order to reach \( \tau \) from \( x_{n_0} \) it takes at least \( \frac{1}{4} \ln \left( \frac{x_{n_0} + c}{2x_{n_0} + c} \right) \) time units. For \( x \in [0, \overline{\tau}] \) let \( \tilde{\tau} = \inf \{ t > 0 \mid R^L_{t_{n_0}} > \overline{\tau} \} \). From the definition of the strategy \( L_{n_0} \) we get that the process has to go through the interval \( [x_{n_0}, \overline{\tau}] \) at least \( n_0 \) times. We get
\[
\delta \tau \geq \frac{n_0 \delta}{i} \ln \left( \frac{i\overline{\tau} + c}{2i\overline{\tau} + c} \right),
\]
and subsequently
\[
\mathbb{E} (e^{-\delta \tau}) \leq \left( \frac{i\overline{\tau} + c}{i(\overline{\tau} - \frac{2x_{n_0}}{c}) + c} \right)^{-n_0} \approx e^{-\frac{n_0}{2} \frac{2\overline{\tau} + c}{c}} \leq \frac{e}{2V(\overline{\tau})},
\]
for \( n_0 \) large enough. Let \( \mathcal{P}_5 \) be the set of paths of \( R^{L_{n_0}} \) with finite \( \overline{\tau} \). Now we define the strategy \( \overline{L} \in \Pi_{\Phi} \), with \( \overline{L} = L_{n_0} \) as long as \( t < \overline{\tau} \), and at \( t = \overline{\tau} \) pay out \( \overline{\tau} \) immediately and distribute the incoming premiums as dividends till the next claim occurrence causes ruin. Again we can write
\[
V_{L_{n_0}}(x) = \mathbb{E} \left( I_{\mathcal{P}_5} \int_0^{t_{L_{n_0}}} e^{-\delta s} \, dL_{n_0,s} \right) + \mathbb{E} \left( I_{\mathcal{P}_5^c} \int_0^{t_{L_{n_0}}} e^{-\delta s} \, dL_{n_0,s} \right)
\]
\[
\leq \mathbb{E} \left( I_{\mathcal{P}_5} \int_0^{t_{L_{n_0}}} e^{-\delta s} \, dL_{n_0,s} \right) + \mathbb{E} \left( I_{\mathcal{P}_5} e^{-\delta \tau} \right) V(\overline{\tau}) + \mathbb{E} \left( I_{\mathcal{P}_5} \int_0^{t_{L_{n_0}}} e^{-\delta s} \, dL_{n_0,s} \right).
\]
Similarly we get
\[
V_{\overline{L}}(x) = \mathbb{E} \left( I_{\mathcal{P}_5} \int_0^{t_{\overline{L}}} e^{-\delta s} \, dL_{\overline{L}} \right) + \mathbb{E} \left( I_{\mathcal{P}_5^c} \int_0^{t_{\overline{L}}} e^{-\delta s} \, dL_{\overline{L}} \right)
\]
\[
\geq \mathbb{E} \left( I_{\mathcal{P}_5} \int_0^{t_{\overline{L}}} e^{-\delta s} \, dL_{\overline{L}} \right) + \mathbb{E} \left( I_{\mathcal{P}_5} \int_0^{t_{\overline{L}}} e^{-\delta s} \, dL_{\overline{L}} \right) + \mathbb{E} (e^{-\delta \tau}) \overline{\tau}.
\]
Because on the sets \( \mathcal{P}_5^c \) and \( \mathcal{P} \) the paths of \( R^{L_{n_0}} \) and \( R^{\overline{L}} \) coincide for \( t < \overline{\tau} \), we arrive at
\[
V_{L_{n_0}}(x) - V_{\overline{L}}(x) \leq \mathbb{E} (e^{-\delta \tau}) (V(\overline{\tau}) - \overline{\tau}) \leq \mathbb{E} (e^{-\delta \tau}) V(\overline{\tau}) \leq \frac{e}{2}.
\]
This finishes the proof since
\[ 0 \leq V(x) - V_T(x) = V(x) - V_{L_n}(x) + V_{L_n}(x) - V_T(x) \leq \epsilon. \]

Finally, the following is a consequence of the proof of Proposition 2.6:

**Lemma 2.10.** Let \( \tau > 0 \) and \( u_1(x) \) be an absolutely continuous supersolution of the HJB equation for all \( x \in [0, \tau] \). If \( L \) is an admissible strategy such that \( R^L_t \leq \tau \) for all \( t \geq 0 \) then \( u_1(x) \geq V_L(x) \) for all \( x \in [0, \tau] \).

These three lemmas imply the following two propositions. The results remind on a similar local characterization of the value function by Shreve et al. [14]. There they also use in an intermediate step the set of constrained controls \( \Pi_\tau \) for solving a dividend maximization problem in a general diffusion setup.

**Proposition 2.11.** If either \( \Lambda(\tau) = 0 \) or \( V'(\tau) = 1 \) for some \( \tau > 0 \) and \( u_1(x) \) is an absolutely continuous supersolution of the HJB equation for all \( x \in [0, \tau] \) then \( u_1(x) \geq V_L(x) \) in \( [0, \tau] \). Hence, if \( L \in \Pi_\tau \) such that \( V_L \) is an absolutely continuous supersolution to the HJB equation for all \( x \in [0, \tau] \) then \( V(x) = V_L(x) \) for all \( x \in [0, \tau] \).

Define for any \( y > 0 \)
\[ U_y(x) = \begin{cases} V(x) & x \leq y, \\ V(y) + x - y & x > y. \end{cases} \]

The following proposition will be the key in the numerical construction of a solution and we will see how it matches some properties of the optimal strategy.

**Proposition 2.12.** (i) If \( U_y \) is a supersolution to the HJB equation in \((y, \infty)\), then \( U_y = V \) in \([0, \infty)\).

(ii) If either \( \Lambda(\tau) = 0 \) or \( V'(\tau) = 1 \) for some \( \tau > 0 \) and there exists \( y < \tau \) such that \( U_y \) is a supersolution of the HJB equation in \((y, \tau]\), then \( U_y = V \) in \([0, \tau]\).

**Proof.** (i) If we prove that \( U_y \) is a supersolution in \( y > 0 \) we immediately have that \( U_y \geq V \) in \([0, \infty)\).

From the definition we have \( U_y(y) = V(y) \) and therefore the supersolution property of \( V \) implies that \( \mathcal{L}_{U_y} \phi(y) \leq 0 \) for an appropriate function \( \phi \). The right-hand derivative in \( y \) is given through
\[ \lim_{x \downarrow y} \frac{U_y(x) - U_y(y)}{x - y} = 1. \]

Remark 2.2 shows that there exists a test function \( \phi \) with the supersolution property if and only if
\[ \lim_{x \downarrow y} \frac{V(x) - V(y)}{x - y} = \lim_{x \downarrow y} \frac{U_y(x) - U_y(y)}{x - y} = 1. \]

But in this case we get \( \phi'(y) = 1 \) showing in addition to \( V \) also \( U_y \) has the supersolution property. To derive \( U_y \leq V \) for all \( x > y \), let \( \epsilon > 0 \) and let \( L \) be an \( \epsilon \)-optimal strategy for initial capital \( y \). For \( x \geq y \) define a strategy \( L_x \) paying out immediately \( x - y \) and subsequently following strategy \( L \). We get
\[ U_y(x) - \epsilon = x - y + V(y) - \epsilon \leq x - y + V_L(y) = V_{L_x}(x) \leq V(x). \]

For (ii) use Proposition 2.11 instead of the general supersolution property. Then the same arguments as above give the desired result.

The following settles the question of differentiability at points switching from the non-pay- to the pay-regime.

**Remark 2.4.** From the proof of Proposition 2.12 (i) and equation (7) of Remark 2.2, we obtain that at points \( y > 0 \) where a barrier strategy with height \( y \) is applied, we have differentiability of the value function: Below \( y \) we use \( V \), in some interval above \( y \) we have \( V \) described by \( U_y \). From Proposition 2.2 and the monotonicity of \( U_y \) we get for \( x < y < x' \) such \((x - y) \geq (x' - y)\),
\[ 1 \leq \frac{V(x) - V(y)}{x - y} = \frac{U_y(x) - U_y(y)}{x - y} \leq \frac{U_y(x') - U_y(y)}{x' - y} \to 1, \]
for \( x' \to y \). This shows that in such change points the left-hand derivative is (by the viscosity solution property) bounded by the right-hand derivative, giving 1 as an upper and lower bound and therefore proving differentiability in these points.
3 Construction of the optimal strategy

3.1 The IDE part of the HJB equation

In intervals where $V'$ exists and is greater than 1 we have to fulfill the second part of the HJB equation (5). Recall that in intervals where it is optimal to do nothing the generator $A$ of the controlled process applied to $V$ gives

$$AV(x) = (c + ix)V'(x) - (\delta + \lambda)V(x) + \lambda \int_0^x V(x - y) \, dF_Y(y).$$

Let us therefore look for a solution of the following integro-differential equation with a given initial condition,

$$0 = (c + ix)f'(x) + \lambda \int_0^x f(x - y) \, dF_Y(y) - (\lambda + \delta)f(x), \quad (18)$$

$$1 = f(0).$$

As for each solution $f(x)$ of (18), $C f(x)$ is again a solution for arbitrary constant $C$, any boundary condition can be fulfilled.

Let $f(x)$ be a solution to (18) and define for some $b \geq 0$

$$V_b(x) = \begin{cases} 
  f(x)/f'(b) & x \leq b, \\
  x - b + V_b(b) & x > b.
\end{cases} \quad (19)$$

An analogue of [13, Lemma 2.49] shows that $V_b$ is equal to the value of the expected discounted dividends when a constant barrier strategy with barrier height $b$ is applied. Hence maximizing $V_b(x)$ over all $b \geq 0$ is equivalent to finding a minimum of $f'(x)$.

We will now prove the existence of a solution of a generalized version of (18). If it is optimal to pay out dividends following a barrier strategy only in a bounded interval ($V' = 1$) and for higher surplus $x > x_0$ it is optimal to pay nothing in some area ($V' > 1$), then we would need a solution to the equation

$$0 = (c + ix)u'(x) - (\delta + \lambda)u(x) + \lambda \int_0^{x-x_0} u(x - y) \, dF_Y(y) + \lambda \int_{x-x_0}^x f(x - y) \, dF_Y(y) \quad (20)$$

$$f(x_0) = u(x_0),$$

where $f : [0, x_0] \to [0, \infty)$ is a given continuous and increasing function. Note that choosing $x_0 = 0$ and taking $u(0) = 1$ as initial condition leads to the existence proof of a solution to (18).

**Lemma 3.1.** Let $x_0 \geq 0$. For a continuous and increasing function $f : [0, x_0] \to [0, \infty)$ there exists a unique, in $(x_0, \infty)$ differentiable and strictly increasing solution $u : (x_0, \infty) \to [0, \infty)$ to (20) with $u(x_0) = f(x_0)$.

**Proof.** For $\epsilon = \frac{c}{2(\delta + \lambda)}$, we will show that there exists a solution with the required properties on $[x_0, x_0 + \epsilon)$ and since $\epsilon$ does not depend on $x_0$ this will establish the existence on $(x_0, \infty)$.

The set of all continuous and increasing functions $u : [x_0, x_0 + \epsilon) \to [0, \infty)$ is denoted by $CI[x_0, x_0 + \epsilon)$, further let for $u \in CI[x_0, x_0 + \epsilon)$,

$$\overline{u}(x) = \frac{(\delta + \lambda)u(x) - \lambda \int_0^{x-x_0} u(x - y) \, dF_Y(y) - \lambda \int_{x-x_0}^x f(x - y) \, dF_Y(y)}{c + ix}.$$

As $u$ and $f$ are continuous, $\overline{u}$ is continuous for $x \geq 0$. Now we define for $u \in CI[x_0, x_0 + \epsilon)$

$$T_u(x) = \int_{x_0}^x \overline{u}(s) \, ds + f(x_0).$$

Because of the monotonicity of $u$ and $f$ and $f(x_0) = u(x_0)$ we get

$$\lambda \int_0^{x-x_0} u(x - y) \, dF_Y(y) + \lambda \int_{x-x_0}^x f(x - y) \, dF_Y(y) \leq \lambda u(x) F_Y(x - x_0) + \lambda f(x_0) F_Y(x) - F_Y(x - x_0)) \leq \lambda u(x).$$
This argument gives the following lower bound for $\pi$

$$0 < \frac{\delta}{c + ix} u(x) \leq \pi(x) \leq \frac{\delta + \lambda}{c + ix} u(x).$$

Here the upper bound follows from the fact that $u$ and $f$ are positive. This implies that $T_u$ is increasing, positive and continuous for $x \in [x_0, x_0 + \epsilon]$. Now for $u_1, u_2 \in CI[x_0, x_0 + \epsilon]$, we get

$$\pi_1(x) - \pi_2(x) = \frac{(\delta + \lambda)(u_1(x) - u_2(x)) - \lambda \int_0^{x-x_0} (u_1(x - y) - u_2(x - y)) dF_Y(y)}{c + ix}$$

$$\leq \frac{1}{c} (||u_1 - u_2|| + \lambda ||u_1 - u_2|| F_Y(x - x_0)) \leq \frac{\delta + 2\lambda}{c} ||u_1 - u_2||,$$

where $|| \cdot ||$ denotes the supremum norm. This implies

$$T_{u_1}(x) - T_{u_2}(x) \leq \epsilon \frac{\delta + 2\lambda}{c} ||u_1 - u_2|| \leq \frac{1}{2} ||u_1 - u_2||.$$

Interchanging $u_1$ and $u_2$ results in $||T_{u_1} - T_{u_2}|| \leq \frac{1}{2} ||u_1 - u_2||$, proving that $T$ is a contraction on $CI[x_0, x_0 + \epsilon]$. Therefore there exists a $u \in CI[x_0, x_0 + \epsilon]$ such that

$$u(x) = \int_{x_0}^x \frac{(\delta + \lambda)u(s) - \lambda \int_0^{s-x_0} u(s - y) dF_Y(y) - \lambda \int_0^{s-x_0} f(s - y) dF_Y(y)}{c + is} ds + f(x_0).$$

Further we have from above that $u'(x) = \pi(x)$ holds everywhere in $[x_0, x_0 + \epsilon]$. This gives the existence of a unique solution to (20) with the required properties on $[x_0, x_0 + \epsilon]$.

\textbf{Remark 3.1.} From the HJB (5) equation we get that at points of differentiability we have that either $V'(x) = 1$ or $L_V(x) = 0$ holds. Lemma 3.1 reveals that differentiability can only be violated at some switching points. Each equation part of (5) has a differentiable solution.

### 3.2 Crucial sets and the optimal strategy

This subsection deals with the construction of a candidate strategy $L^*$ for the optimal one. Although it is not possible to directly show that $V^{L^*}$ is a supersolution of (5) and verify its optimality with Proposition 2.6, it is possible to prove that $V^{L^*} = V$ via a fixed point argument, proving the optimality of the strategy $L^*$. Actually a full characterization of the value function is needed to obtain the correct solution with the construction of $L^*$ (another solution of (5) with an arbitrary initial value for the definition of $L^*$ would not lead to the solution of the maximization problem).

The following three sets will play a crucial role in the definition of the optimal strategy.

- $A = \{x \in [0, \infty) \mid \Lambda(x) = 0\}$,
- $B = \{x \in (0, \infty) \mid V'(x) = 1 \text{ and } \Lambda(x) < 0\}$,
- $C = (A \cup B)^\circ$.

Let us identify some properties of these sets.

\textbf{Proposition 3.2.} 1. $B$ is a left-open set, i.e. for each $x \in B$ $\exists \delta > 0$ such that $(x - \delta, x] \subset B$.

2. $A$ is a closed set.

3. If $(x_0, \pi) \subset B$ and $x_0 \notin B$ then $x_0 \in A$.

4. $\exists \hat{x}$ such that $(\hat{x}, \infty) \subset B$.

5. $C$ is a right-open set, i.e. for each $x \in C$ $\exists \delta > 0$ such that $[x, x + \delta) \subset C$.

6. $A \cap B \neq \emptyset$.
Proof. 1. The idea is as follows: if for sufficiently small $h > 0$ we are able to show that $U_{x-h}$ is a supersolution in $(x-h, x]$, then we get from Proposition 2.1(ii) that $U_{x-h} = V$ in $[0, x]$, and hence $V' = 1$ in $(x-h, x]$ implying $(x-h, x] \subset B$.

Let $y \in (x-h, x)$ and recall from the definition of $B$ that $L_V(x) < 0$.

\[
\begin{align*}
\mathcal{L}_{U_{x-h}}(y) &= (c + iy) - (\delta + \lambda)(y - x + h + V(x-h)) + \lambda \int_{y}^{x} U_{x-h}(y - z) dF_V(z) \\
&\leq (c + iy)V'(x) - (\delta + \lambda)V(x) + (\delta + \lambda)(V(x) - (y - x + h + V(x-h))) + \lambda \int_{y}^{x} U_{x-h}(y - z) dF_V(z) \\
&\leq (c + iy)V'(x) - (\delta + \lambda)V(x) + (\delta + \lambda)(V(x) - V(x-h)) + \lambda \int_{y}^{x} U_{x-h}(y - z) dF_V(z) \\
&\leq \mathcal{L}_V(x) + (\delta + \lambda)(V(x) - V(x-h)) < 0.
\end{align*}
\]

The last step holds for $h$ small enough because of the continuity of $V$, $y < x$ and the following estimates,

\[
\begin{align*}
V(y - z) &\geq y - z - x + h + V(x-h), \quad \text{for } y - z \geq x-h, \\
\int_{y}^{x} U_{x-h}(y - z) dF_V(z) &= \int_{y-x+h}^{x} (y - z - x + h + V(x-h))dF_V(z) + \int_{y-x+h}^{y} V(y - z) dF_V(z) \\
&\leq \int_{y}^{x} V(x-z) dF_V(z).
\end{align*}
\]

We proved that $U_{x-h}$ is indeed a supersolution in $(x-h, x]$ and therefore the statement holds.

2. Because $\Lambda$ is continuous in $x$ and $\Lambda(x) \leq 0$ for all $x \in [0, \infty)$, the region where it equals 0 is closed. Assume that there is some $x_0$ such that $\Lambda(x_0) > 0$ then because of the continuity there is a $x_1 > x_0$ such that $\Lambda > 0$ in $[x_0, x_1)$. Let $y \in (x_0, x_1)$ such that $V'(y)$ exists. Because $V' \geq 1$ we get

\[
\mathcal{L}_V(y) = (c + iy)V'(y) - (\delta + \lambda)V(y) + \lambda \int_{0}^{y} V(y - z) dF_V(z) \\
\geq (c + iy) - (\delta + \lambda)V(y) + \lambda \int_{0}^{y} V(y - z) dF_V(z) = \Lambda(y) > 0,
\]

which is a contradiction to the fact that $V$ is a viscosity supersolution to the HJB equation (5).

3. First we deal with the case $x_0 = 0$. We know that $V(0) \geq \frac{c}{\delta + \lambda}$. This will also be an upper bound, implying that $\Lambda(0) = 0$ $(x_0 \in A)$. Because $[0, x] \subset B$ we have that $V(x) = x + V(0)$ in $[0, x]$. For $x \in (0, x]$ we have from Lemma 2.9 that $V(0) = \sup_{L \in \Pi_x} V_L(0)$.

Let $L \in \Pi_x$, the time of the first claim occurrence be $\tau_1$ and its size $Y_1$. For all $t \leq \tau_1$ we have $L_t = \int_{0}^{t}(c+ix)ds$, $L_t \leq ct + i \int_{0}^{t} R_s^L ds \leq (c+ix)t$ due to $L \in \Pi_x$ and the definition of an admissible strategy. We get the obvious upper bound

\[
V_L(0) = \mathbb{E} \left( \int_{0}^{T_1} e^{-\delta s} dL_s + e^{-\delta T_1}V \left( \int_{0}^{T_1} (cT_1 + i \int_{0}^{T_1} R_s^L ds - L_{T_1} - Y_1) \right) \right) \\
\leq \mathbb{E} \left( \int_{0}^{T_1} e^{-\delta s}(c+ix)ds \right) + \mathbb{E} \left( e^{-\delta T_1}V(x-y) \right) \\
= \int_{0}^{x} \lambda e^{-\lambda t} \left( c+ix \right) dt + \int_{0}^{x} \lambda e^{-\lambda t} \int_{0}^{x} V(x-y) dF_V(y) dt \\
= \int_{0}^{x} e^{-\lambda t} \left( c+ix \right) + \lambda \int_{0}^{x} V(x-y) dF_V(y) dt.
\]

Using $V(x) = x + V(0)$ in the specific area, we arrive at

\[
V(0) = \lim_{x \to 0} \frac{1}{\delta + \lambda} \left( c+ix \right) + \lambda V(0)F(x) + \lambda x F(x) - \lambda \int_{0}^{x} y F(y) = \frac{c}{\delta + \lambda},
\]

20
which proves the statement for \( x_0 = 0 \).

Now we deal with the case \( x_0 > 0 \) following [3]. If \( V'(x_0) = 1 \) and \( x_0 \notin \mathcal{B} \) we get that \( \Lambda(x_0) = 0 \) and therefore by definition \( x_0 \notin \mathcal{A} \). We have \( \lim_{x \to x_0} V'(x) - V(x_0) = 1 \). Suppose \( \lim_{x \to x_0} \frac{V(x) - V(x_0)}{x - x_0} = \gamma > 1 \. Then we have from Remark 2.2 for all \( 1 < p \leq q \)

\[
\max\{1 - p, (c + ix_0)p - (\delta + \lambda)V(x_0) + \lambda \int_0^{x_0} V(x_0 - y) \, dF_Y(y)\} \geq 0,
\]

which implies

\[
(c + ix_0)p - (\delta + \lambda)V(x_0) + \lambda \int_0^{x_0} V(x_0 - y) \, dF_Y(y) \geq 0.
\]

The limit \( p \to 1 \) gives \( \Lambda(x_0) \geq 0 \) which implies \( \Lambda(x_0) = 0 \).

Now we assume \( \lim \inf_{x \to x_0} \frac{V(x) - V(x_0)}{x - x_0} = 1 \). There is a sequence \( \{x_n\} \) with \( x_n \to x_0 \) such that \( \lim_{n \to \infty} V'(x_n) = 1 \). Choose a sequence \( \{h_n\} \) with \( h_n \downarrow 0 \) such that \( \lim_{n \to \infty} \frac{V(x_0) - V(x_0 - h_n)}{h_n} = 1 \). Take \( a_n = \frac{V(x_0) - V(x_0 - h_n)}{h_n} - 1 \) and let \( A_n \) denote the set of all \( x \in [0, h_n] \) such that \( V' \) exists and \( V'(x) \geq 1 + 2a_n \). Because of the inequalities for the first derivative, see Proposition 2.2, we can assume \( a_n \geq 0 \). If for some \( n \) we would have \( a_n = 0 \) we get \( V(x_0) - V(x) = x_0 - x \) for \( x \in [x_0 - h_n, x_0] \) and therefore \( V'(x_0) = 1 \). Therefore assume \( a_n > 0 \), and we can write by the absolute continuity, \( |A_n| \leq h_n \), and \( A_n' = [0, h_n]\backslash A_n \)

\[
a_n + 1 = \frac{\int_{A_n} V'(z) \, dz + \int_{A_n'} V'(z) \, dz}{h_n} \geq \frac{|A_n|(1 + 2a_n) + (h_n - |A_n|)}{h_n}.
\]

This gives the estimates \( |A_n| \leq \frac{h_n}{2a_n} \to 0 \). So we can choose a sequence \( x_n \to x_0 \) with \( 1 \leq V'(x_n) \leq 1 + 2a_n \) such that \( V'(x_n) \) exists. In the end we get \( \lim_{n \to \infty} V'(x_n) = 1 \).

If there is a subsequence \( x_{n_j} \to x_0 \) with \( V'(x_{n_j}) > 1 \) implying \( \Lambda(x_{n_j}) = 0 \) we would have \( \Lambda(x_0) = 0 \) because \( \mathcal{A} \) is a closed set. Suppose \( V'(x_n) = 1 \) for all \( n \in \mathbb{N} \) and \( \Lambda(x_0) < 0 \). Then we can find an \( x_n \) close enough to \( x_0 \) (\( \Lambda \) is continuous) such that \( U_{x_n} \) is a supersolution for \( x \in [x_n, x_0] \) but Proposition 2.12 yields that \( U_{x_n} = V \) in \( [0, \bar{\tau}] \). This gives a contradiction because \( V \) would be differentiable at \( x_0 \).

\[
\mathcal{L}U_{x_n}(x) = (c + ix) - (\delta + \lambda)U_{x_n}(x) + \lambda \int_0^x U_{x_n}(x - y) \, dF_Y(y)
\]
\[
\leq (c + ix_0) - (\delta + \lambda)U_{x_n}(x) + \lambda \int_0^x (V(x_0 - y) - dF_Y(y)
\]
\[
= \Lambda(x_0) + (\delta + \lambda)\int (x - x_n + V(x_n))
\]
\[
\leq \Lambda(x_0) + (\delta + \lambda)(V(x_0) - V(x_n)) < 0.
\]

The last inequality holds due to the continuity of \( V \) for \( n \) large enough. This proves the third point.

4. We want to show that for \( y > 0 \) large enough \( U_y \) is a supersolution for all \( x \in (y, \infty) \). We already have \( U'_y = 1 \) in this interval, it is left to show that \( \mathcal{L}U_y(x) < 0 \). We have

\[
\mathcal{L}U_y(x) = (c + ix) - (\delta + \lambda)(x - y + V(y)) + \lambda \int_0^x U_y(x - z) \, dF_Y(z)
\]
\[
\leq (c + ix) - (\delta + \lambda)(x - y + V(y)) + \lambda \int_0^x (V(x_0 - y) + V(x))
\]
\[
= (c + ix) - (\delta + \lambda)(x - y + V(y)) + \lambda(x - y + V(y))
\]
\[
= c + (i - \delta)(y - V(y)) \leq c + (i - \delta)y - \frac{c}{\delta + \lambda} < 0.
\]

This holds for every \( x \in (y, \infty) \) if \( y \) is large enough, because \( U_y \) is an increasing function and

\[
\int_0^x U_y(x - z) \, dF_Y(z) \leq U_y(x),
\]
\[
y + \frac{c}{\delta + \lambda} \leq V(y).
\]
5. For some $x \in \mathcal{C}$ we have $\Lambda(x) < 0$. Because of continuity we get the existence of a $\delta > 0$ such that $[x, x + \delta) \subset \mathcal{A}^c$. If there would be some $x_1 \in \mathcal{B}$ within this interval we would derive the existence of an $x_0 \in \mathcal{A}$ smaller than $x_1$ such that $(x_0, x_1] \subset \mathcal{B}$, but because $x \notin \mathcal{B}$ this $x_0$ also has to be in the interval $(x, x + \delta)$. Therefore we have $[x, x + \delta) \subset \mathcal{B}^c$ and $(x, x + \delta) \subset \mathcal{C}$.

6. The statement follows from the third and fourth point.

At this stage we are able to define the optimal strategy.

**Definition 3.1.** The optimal strategy $L^*$ is stationary, i.e. it depends only on $x = R^L_{t-} \geq 0$, and is given as follows:

- If $x \in \mathcal{A}$, everything exceeding $x$ is paid out immediately as dividend (with rate $c + ix$).
- For $x \in \mathcal{B}$, we know from Proposition 3.2 that there is a $x_1 \in \mathcal{A}$ such that $(x_1, x] \subset \mathcal{B}$, and dividends are paid with the amount $x - x_1$.
- For $x \in \mathcal{C}$ no dividends are paid.

From [3] one knows that the strategy as defined above is admissible. The following proposition shows that this band strategy is indeed optimal.

**Proposition 3.3.** The strategy $L^*$ defined in Definition 3.1 is optimal, i.e. $V(x) = V_{L^*}(x)$ for all $x \geq 0$.

**Proof.** From Proposition 3.2 we know that there exists some $\hat{x} = \inf \{x \mid (x, \infty) \subset \mathcal{B}\}$. We want to define a contraction map on the set of all functions $f : \mathbb{R} \to [0, \infty)$ with $f(x) = 0$ for $x < 0$ and $f(x) = x - \hat{x} + f(\hat{x})$ for $x > \hat{x}$ which are continuous on $\mathbb{R}^+$. The used distance measure is $d(f_1, f_2) = \max_{x \geq 0} |f_1(x) - f_2(x)|$. The operator $T$ is defined as follows,

$$T_f(x) = \mathbb{E} \left( \int_0^{\tau_1} e^{-\delta s} dL^*_s + e^{-\delta \tau_1} f \left( e^{\tau_1} (x + \int_0^{\tau_1} (c - l^*_s) e^{-is} ds) - \Delta L^* - Y_1 \right) \right),$$

where $\tau_1$ denotes the time of the first claim occurrence and $Y_1$ its size. Notice the similarity to the dynamic programming principle (4) with $R^L_{t-} = e^{it} (x + \int_0^t (c - l^*_s) ds) - \Delta L^*$ where $l^*$ denotes the density of the absolutely continuous part of $L^*$. From Definition 3.1 we have that $l^* = 0$ for $x \in \mathcal{B} \cup \mathcal{C}$ and $l^* = c + ix$ for $x \in \mathcal{A}$.

One gets

$$|T_{f_1} - T_{f_2}| \leq \mathbb{E} \left( e^{-\delta \tau_1} f_1 \left( e^{\tau_1} (x + \int_0^{\tau_1} (c - l^*_s) e^{-is} ds) - \Delta L^* - Y_1 \right) \right)$$

$$- f_2 \left( e^{\tau_1} (x + \int_0^{\tau_1} (c - l^*_s) e^{-is} ds) - \Delta L^* - Y_1 \right) \right)$$

$$\leq \frac{\lambda}{\delta + \lambda} \max_{x \geq 0} |f_1(x) - f_2(x)|,$$

therefore $T$ is a contraction and has a unique fixed point. The definition of $L^*$ ensures that $T_f$ is in the same space as $f$. Clearly $V_{L^*}$ is a fixed point because of the dynamic programming principle and the definition of $L^*$. Now we are going to show that $V$ is also a fixed point which gives $V = V_{L^*}$.

We start with $x \in \mathcal{A}$, then

$$TV(x) = \mathbb{E} \left( \int_0^{\tau_1} (c + ix) e^{-\delta s} ds + e^{-\delta \tau_1} V(x - Y_1) \right)$$

$$= \frac{1}{\delta + \lambda} \left( (c + ix) + \lambda \int_0^x V(x - y) dF_V(y) \right) = V(x),$$

because $\Lambda(x) = 0$ for $x \in \mathcal{A}$.

Next, we look at $x \in \mathcal{B}$. Let $x_1$ such that $(x_1, x] \subset \mathcal{B}$ and $x_1 \in \mathcal{A}$. We get from the definitions of $L^*$ and $\mathcal{B}$,

$$TV(x) = x - x_1 + TV(x_1) = x - x_1 + V(x_1) = V(x).$$

22
Finally, we know that $C$ is a right-open set. Therefore some $x_1$ exists such that $[x, x_1) \subseteq C$ and $x_1 \notin C$. Denote
\[ x_t = e^{it} \left( x + c \int_0^t e^{-is} ds \right), \]
and let $t_1$ such that $x_{t_1} = x_1$. Because $V$ is a differentiable solution to $(c + iz)V'(z) - (\delta + \lambda) V(z) + \lambda \int_0^z V(z - y) \, dF(y) = 0$ for $z \in (x, x_1)$, $\frac{d}{dt} x_t = c + i x_t$ and
\[ \frac{d}{dt} e^{-(\delta + \lambda)t} V(x_t) = -(\delta + \lambda) e^{-(\delta + \lambda)t} V(x_t) + e^{-(\delta + \lambda)t} (c + i x_t) V'(x_t). \]
So we get
\[ T(V)(x) = E \left( I_{\{t_1 \geq t\}} e^{-\delta t_1} V(x_1) + E \left( I_{\{t_1 < t\}} e^{-\delta t_1} V(x_{t_1} - Y_1) \right) \right) = e^{-(\delta + \lambda)t_1} V(x_1) + \int_0^{t_1} e^{-(\delta + \lambda)t} \lambda \int_0^{x_1} V(x_1 - y) \, dF(y) \, dt \]
\[ = e^{-(\delta + \lambda)t_1} V(x_1) + \int_0^{t_1} e^{-(\delta + \lambda)t} \left( (\delta + \lambda)V(x_1) - (c + i x_t)V'(x_t) \right) dt \]
\[ = e^{-(\delta + \lambda)t_1} V(x_1) + V(x) - e^{-(\delta + \lambda)t_1} V(x_1) = V(x). \]

From Remark 2.4 we know that $V'$ can not have any downward jumps and further that (20) has a differentiable solution. Therefore the only possibility of not being differentiable is at points where the optimal strategy changes from paying a lump sum to paying no dividends.

The similarity to the optimal strategy for the case $i = 0$ as it is dealt with in [3] and [13] allows us to use an algorithm from [13] to determine the value function piecewise. As mentioned in Section 3.1 and because of the construction of the band strategy there is a close relation to barrier strategies. For small initial capital the first thing to do is to find a local optimal barrier, i.e. find the smallest point in the set $A$ denoted by $x_0$. Notice that it is possible that $0 \in A$. Let $f_0$ be the solution of (18) and choose the smallest point in $A$ as $x_0 = \sup\{x \geq 0 \mid f_0(x) = \inf_{y \geq 0} f_0'(y)\}$. Then define
\[ v_0(x) = \begin{cases} f_0(x)/f_0'(x_0) & x \leq x_0, \\ x - x_0 + f_0(x_0)/f_0'(x_0) & x > x_0. \end{cases} \]

If $v_0$ fulfills the HJB equation (5) we are done, if not the solution is constructed recursively: In the $n$th step ($n \geq 1$), find some interval belonging to $B$ of the form $(x_n, a)$ (cf. Proposition 3.2). Then it is possible that some adjoining interval $[a, x_{n+1})$ belongs to the set $C$; then it is necessary to calculate a solution to (20). The points $a$ and $x_{n+1}$ are determined in the following way. For given $v_n(x)$ and $x_n$, let $f_{n+1}(x; y)$ be a solution of (20) for $x \geq y$ and equal to $v_n(x)$ for $x < y$. We have to find the smallest $y > x_n$ such that $f_{n+1}'(x; y) = 1$ for some $x > y$.
\[ a = \inf \{ y \geq x_n \mid \inf_{z > y} f_{n+1}'(z, y) = 1 \}. \]

If $a$ is chosen too small or too large then the derivative of $f_{n+1}'(x; \cdot)$ will either take a minimum greater than 1 or smaller than 1. Due to Proposition 2.2 and the fact that $V'$ can not have downward jumps a wrong choice would not lead to a solution of the maximization problem.

Then we obtain $x_{n+1} := \sup \{ x \geq a \mid f_{n+1}'(x, a) = 1 \}$ and
\[ v_{n+1}(x) := \begin{cases} f_{n+1}(x, a), & x \leq x_{n+1}, \\ x - x_{n+1} + f_{n+1}(x_{n+1}, a), & x > x_{n+1}. \end{cases} \]

If $v_{n+1}(x)$ fulfills (5) we have constructed the value function, otherwise we restart the procedure. Because of Proposition 3.2 this algorithm terminates.
4 Examples

4.1 $\text{Exp}(\alpha)$ distributed claim amounts

In the first example consider exponential claim amounts with $F_Y = 1 - e^{-\alpha y}$. We will see that in the case $0 < i < \delta$ a barrier strategy is optimal, an analogous result for $i = 0$ was first shown in [7]. To find an element of $A$ we need to solve $\Lambda(x) = 0$, because of the properties of the set $B$ some of these elements are lower boundaries of subsets of $B$. Looking for a solution to $\Lambda(x) = 0$ we observe that we have to solve

$$V(x) = \frac{c + ix}{\delta + \lambda} - \frac{\lambda}{\delta + \lambda} e^{-\alpha x} \int_0^x V(y) e^{\alpha y} dy. \quad (21)$$

If a point $a \in A$ is a boundary point of a connection component of $B$ we have $V'(a+) = 1$. From $V' \geq 1$ and the fact that $V'$ cannot have downward jumps (see Remark 2.4) we get $V'(a) = 1$. Therefore, by $B \neq \emptyset$ we can additionally use the condition $V'(x) = 1$ for at least one element of $A$. From (21) and $V' = 1$ we get,

$$1 = V'(x) = \frac{i}{\delta + \lambda} - \frac{\alpha^2 \lambda}{\delta + \lambda} e^{-\alpha x} \int_0^x V(y) e^{\alpha y} dy + \frac{\alpha \lambda}{\delta + \lambda} V(x).$$

Using (21) again to eliminate the integral we derive,

$$\frac{\delta + \lambda}{\alpha} - \frac{i}{\alpha} = c + ix - \delta V(x).$$

Since $i < \delta$ and $V(x) \geq x + \frac{c}{\alpha + \delta}$ (Proposition 2.1) we further have that $ix - \delta V(x)$ is decreasing. There exists at most one positive point on the real axis which fulfills these conditions. This is equivalent to the statement that a barrier strategy $b^*$ is the optimal one in the case of $\text{Exp}(\alpha)$ distributed claim amounts. In the following we identify the case $b^* > 0$. The case of an optimal barrier equal to zero is then treated in Section 4.1.2.

4.1.1 The case $b^* > 0$

For the determination of the optimal barrier we can use some results from [10], a by-product we can show why only the case $i < \delta$ makes sense mathematically. The structure of a constant barrier strategy is as follows. Given a barrier at level $b$, all surplus above this level will be immediately paid out as dividend. We denote the expected discounted dividends for a barrier $b$ with $V_b(x)$. Assuming differentiability of $V_b(x)$ we get the following well-known IDE (see [10]), for $x < b$

$$0 = (c + ix)V'(x) + \lambda \int_0^x V(y-x) dF_Y(y) - (\lambda + \delta)V(x), \quad (22)$$

$$1 = V'_b(b). \quad (23)$$

From the nature of a barrier strategy we have for $x > b$

$$V_b(x) = x - b + V_b(b).$$

Because (22) is homogeneous and linear in $V$ we can look for a solution $f$ of it with a modified initial condition $f(0) = 1$. By scaling we get that $V_b = f(x)/f'(b)$ for $0 \leq x \leq b$. Following [10] we have to solve

$$0 = (c + ix)f''(x) + (\alpha(c + ix) + i - (\delta + \lambda))f'(x) - \alpha f(x) = 0,$$

$$1 = f(0),$$

$$0 = c f'(0) - (\delta + \lambda)f(0).$$

The general solution is of the form

$$f(x) = e^{-\alpha x} \left( x + \frac{c}{i} \right)^{(\lambda+i)/i} \left( b_1 F \left( 1 + \frac{\delta - \lambda}{i}, 1 + \frac{\lambda + \delta}{i}, \alpha(u + \frac{c}{i}) \right) + b_2 U \left( 1 + \frac{\delta - \lambda}{i}, 1 + \frac{\lambda + \delta}{i}, \alpha(u + \frac{c}{i}) \right) \right),$$

24
where \( B_1 \) and \( B_2 \) are constants determined by the boundary conditions and \( F \) and \( U \) are confluent hypergeometric functions of the first and second kind, respectively. Because maximizing \( V_0 \) is equivalent to minimizing \( f' \) we take a look on the asymptotics of \( f \) and \( f' \). From [1] we have

\[
F(a, b, z) \sim \frac{\Gamma(b)}{\Gamma(a)} e^{z} z^{a-b} (1 + O(|z|^{-1})),
\]

\[
U(a, b, z) \sim z^{-a} (1 + O(|z|^{-1})).
\]

So we get

\[
f(x) \sim \frac{\Gamma(1 + \frac{\delta+b}{2})}{\Gamma(1 + \frac{\delta}{2})} B_1 e^{x} e^{\alpha x} \left( x + \frac{c}{x} \right)^{\frac{x}{x}} (1 + O((\alpha(x + \frac{c}{x}))^{-1})).
\]

We can use the same asymptotics to obtain the behaviour of \( f'(x) \) for large \( x \) and it suffices to consider the terms in connection to \( F(a, b, z) \). Therefore we get

\[
f'(x) \sim B_1 \frac{\delta + \lambda}{\delta} \frac{\Gamma(1 + \frac{\delta+b}{2})}{\Gamma(1 + \frac{\delta}{2})} e^{x} \alpha x \left( x + \frac{c}{x} \right)^{\frac{x}{x}} (1 + O((\alpha(x + \frac{c}{x}))^{-2})),
\]

with some constant \( K \). Furthermore

\[
\lim_{x \to \infty} f'(x) = \begin{cases} 0, & \delta < i, \\ \infty, & \delta > i, \\ \text{const.}, & \delta = i. \end{cases}
\]

and as a consequence for a fixed argument \( x \)

\[
\lim_{b \to \infty} V_0(x) = \lim_{b \to \infty} \frac{f(x)}{f'(b)} = \begin{cases} \infty, & \delta < i, \\ 0, & \delta > i, \\ \frac{f(x)}{\text{const.}}, & \delta = i. \end{cases}
\]

Since \( V_0(x) \leq V(x) \) the value function is unbounded for \( i > \delta \) and does not fulfill \( \lim_{b \to \infty} V_0(x) \) for \( \delta = i \). Therefore only the case \( \delta > i \) is interesting and leads to a well-formulated dividend maximization problem. If \( b^* > 0 \), then calculate \( f \) and determine \( b^* = \text{argmax} \{ f'(b) \mid b > 0 \} \) numerically. Then

\[
V(x) = V_b(x) = \begin{cases} f(x)/f'(b^*) & 0 \leq x \leq b^*, \\ x - b^* + f(b^*)/f'(b^*) & x > b^*. \end{cases}
\]

As an illustration Figure 1 shows the value function when the optimal barrier strategy with height \( b^* = 4.41 \) is applied together with the two linear bounds from Proposition 2.1 (which are obviously not tight). The chosen parameters are \( \alpha = 2, \lambda = 2, i = 0.05, \delta = 0.1 \) and \( c = 2.5 \).

### 4.1.2 The case \( b^* = 0 \)

We need to determine parameter settings for which \( b^* = 0 \) is optimal. For \( b^* = 0 \), \( V(x) = V_0(x) = x + \frac{c}{\alpha x} \).

Because in this case \( V_0'(x) = 1 \) for \( x \geq 0 \), we only have to check when

\[
(c + ix) - (\delta + \lambda)V_0(x) + \lambda \int_0^x V_0(x-y)\alpha e^{-\alpha y} dy \leq 0
\]

holds. Evaluating this equation, it turns out that for

\[
Z(x) := (\delta + \lambda - \alpha \delta + x(i - \delta)\alpha(\delta + \lambda) - \lambda(\delta + \lambda - \alpha \delta),
\]

we have to check when \( Z(x) \leq 0 \) for all \( x \geq 0 \). Further

\[
Z'(x) = (i - \delta) - \frac{\lambda}{\delta + \lambda} e^{-\alpha x} (\delta + \lambda - \alpha \delta).
\]
If $\delta + \lambda \geq ac$ we have $Z' \leq 0$ for all $x \geq 0$ and $Z(0) = 0$ is a maximum of $Z$. Therefore $V_0$ fulfills the HJB equation and $V_0 = V$.

Because $Z$ is continuously differentiable we can discuss the behaviour of this function in the usual way. If at the maximum of $Z$ attained in a point $\hat{x}$ the value $Z(\hat{x}) \leq 0$, $V_0$ is the value function (if $Z(\hat{x}) > 0$ we have to look for a positive optimal barrier described as in Section 4.1.1).

We get that $Z'$ has a root in $\hat{x} = \frac{1}{\alpha} \log \left( \frac{\lambda(ac-(\delta+\lambda))}{(\delta-i)(\delta+\lambda)} \right) \in \mathbb{R}$. Additionally we have $Z''(\hat{x}) = \alpha(i - \delta) < 0$, such that

$$Z(\hat{x}) = \frac{\lambda(ac-(\delta+\lambda)) - (\delta - i)(\delta + \lambda) + (i - \delta)(\delta + \lambda) \log \left( \frac{\lambda(ac-(\delta+\lambda))}{(\delta-i)(\delta+\lambda)} \right)}{\alpha(\delta + \lambda)}$$

is a global maximum. Note $\hat{x} > 0 \iff \lambda(ac-(\delta+\lambda)) > (\delta-i)(\delta+\lambda)$. Because $Z''(x) = -\frac{e^{-ax}a\lambda(ac-(\delta+\lambda))}{\delta+\lambda} < 0$ we have that $Z''(x) < 0$ for $x > \hat{x}$ and $Z'(x) > 0$ for $x < \hat{x}$. This has the consequence that if $\hat{x} \leq 0$, $Z(0) = 0$ is again a maximum for positive arguments such that $V_0 = V$. In the case $\hat{x} > 0$ $Z(\hat{x}) < 0$ has to be checked. Summarizing, $V_0(x) = V(x)$ for $x \geq 0$ if one of the following cases holds for the set of parameters:

1. $ac \leq (\delta + \lambda)$,
2. $ac > (\delta + \lambda) \land \frac{ac-(\delta+\lambda)}{\delta-i} \leq \frac{(\delta-i)(\delta+\lambda)}{\lambda}$,
3. $ac > (\delta + \lambda) \land \frac{ac-(\delta+\lambda)}{\delta-i} > \frac{(\delta-i)(\delta+\lambda)}{\lambda}$ and 

$$\lambda(ac-(\delta+\lambda)) - (\delta - i)(\delta + \lambda) \leq (\delta - i)(\delta + \lambda) \log \left( \frac{\lambda(ac-(\delta+\lambda))}{(\delta-i)(\delta+\lambda)} \right).$$

If none of these cases holds, calculate $V = V_0$, as described in Section 4.1.1.

### 4.2 Gamma$(2, \gamma)$ distributed claim amounts

In this section we will identify an explicit example where a band strategy is optimal. In contrast to the case $i = 0$ (of [3] and [13]) an explicit solution to

$$(c + ix)f'(x) - (\delta + \lambda)f(x) + \lambda \int_0^x f(x - y) d_x(y) dy = 0$$

is not available, where $d_x(y) = y\gamma^2 e^{-\gamma y}$ denotes the Gamma$(2, \gamma)$ density function. Therefore we need numerical solutions to (18) and (20) for applying the algorithm presented in [13]. A natural approach is
to use the contraction argument from Lemma 3.1 for determining a numerical solution but that turns out to be too time consuming and inaccurate. So here we implement another approach to obtain a reasonably accurate solution of (20).

Assume that the value function is determined up to a point \( x_0 \). Following the algorithm from [13] (see Section 3.2) we have to calculate \( f_{x+1}(x; y) \) as a solution to (20) with \( x_0 \) replaced by \( y \). In terms of the algorithm the initial condition is given by \( v_0(y) = f_{x+1}(y; y) \). First we fix a step width \( h > 0 \) and choose a set of points \( \{x_j\} \) with \( y_k = y + kh \) then we define piecewise linear functions \( \{\omega_k(x)\} \) such that \( \omega_k(y_{k-1}) = 0 \), \( \omega_k(y_k) = 1 \), \( \omega_k(y_{k+1}) = 0 \) and \( \omega_k(x) = 0 \) for \( x \notin [y_{k-1}, y_k + 1] \). Let the sequence \( \{u_k\} \) denote the unknown values of a solution to (20) at the points \( y_k \). The numerical solution we are looking for is of the form

\[
u(x) = \sum_{k=0}^{K} u_i \omega_k(x).
\]

Plugging \( u(x) \) into (20) and evaluating this expression at every \( y_k \) leads to a linear system of equations for the unknowns \( u_k \).

Finally we give a concrete example for a situation where a band strategy is optimal. Choose the parameters by \( \lambda = 10 \), \( \delta = 0.1 \), \( \gamma = 1 \), \( c = 21.4 \) (cf. [3]) but now with a positive interest rate \( i = 0.02 \). First observe that if we look at a solution to (18), the derivative is minimized in zero. On the other hand \( x + \frac{a}{x} \) does not fulfill (5) on \( \mathbb{R}^+ \). Therefore we have to choose \( x_0 = 0 \) and apply the numerical method presented above. We get that the sets \( A, B \) and \( C \) are given by

\[
A = \{0, 12.96\},
B = (0, 0.96) \cup (12.96, \infty)
C = [0.96, 12.96].
\]

A sample path of the reserve process controlled by the optimal strategy \( L^* \) is illustrated in Figure 2. Starting with initial capital \( x \in B \) the amount \( x - x_1, x_1 = 12.96 \), is immediately paid out as dividend, this lump sum payment is marked as the bold dashed line with an arrow at its end. Then up to the first claim occurrence which takes the process to \( C \), dividends are paid continuously at a rate \( c + ix_1 \). In the set \( C \) there are no actions on the reserve process such that it increases again to \( x_1 \) and stays there till the second claim happens, there again dividends are paid with intensity \( c + ix_1 \). Jumping into set \( B \) the reserve is immediately reduced to the next point in the set \( A \) which is \( x_0 = 0 \). The process stays at this level, i.e. dividends are paid with intensity \( c \), till ruin caused by the third claim occurs. Figure 3 shows the value function for \( i = 0.02 \) in comparison to the value function with \( i = 0 \) (dashed line, as calculated in [3]). It can be observed that for low initial capital both follow the same strategy, but from 0.96 on the case with positive interest rate dominates the other one.

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References


Figure 2: Path of $R^L$

Figure 3: Value function for $\lambda = 10$, $\delta = 0.1$, $\gamma = 1$, $c = 21.4$, $i = 0.02$ and $\dot{i} = 0$


