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Tangential-Displacement and Normal-Normal-Stress Continuous Mixed Finite Elements for Elasticity
TANGENTIAL-DISPLACEMENT AND NORMAL-NORMAL-STRESS CONTINUOUS MIXED FINITE ELEMENTS FOR ELASTICITY

JOACHIM SCHÖBERL AND ASTRID SINWEL

Abstract. In this paper we introduce new finite elements to approximate the Hellinger Reissner formulation of elasticity. The elements are the vector valued tangential continuous Nédélec elements for the displacements, and symmetric, tensor valued, normal-normal continuous elements for the stresses. These elements do neither suffer from volume locking as the Poisson ratio approaches $\frac{1}{2}$, nor suffer from shear locking when anisotropic elements are used for thin structures. We present the analysis of the new elements, discuss their implementation, and give numerical results.

1. Introduction

Finite element simulation of mechanical problems has a long history, and the theory is well developed. But, as material parameters or some geometric dimensions become bad, the primal method may lead to very poor results. This effect is known as locking. Here, many non-standard methods such as mixed variational principles have been introduced [14].

We consider the Hellinger Reissner formulation, where the displacement vector as well as the stress tensor are approximated as independent unknowns. There are essentially the two possibilities to apply the derivatives either to the displacements, or to the stresses. The first one is easy to discretize, but leads usually to the standard method suffering from locking. The second one needs stress tensor elements with continuous normal vectors, which are more difficult to construct [9, 1, 7, 10]. They may lead to better approximation properties. An alternative are elements with weak symmetry [6, 26, 27, 12, 8]. In the present paper, we develop a formulation, which is in between these two concepts. The obtained finite elements have good approximation properties and are easy to implement.

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We propose elements which have continuous tangential components for the displacement vector $u$, and continuous normal-normal component of the stress tensor $\sigma$. The lowest order stable elements for a triangle is drawn in Figure 1. The displacement element has one degree of freedom for the tangential component along edges. This is exactly the lowest order Nédélec element. The normal component of the normal stress vector is approximated as linear function along the edges. There are exactly three dofs for each edge which are necessary to fix the rigid body motions in 2D: The displacement is prescribed in tangential direction, and the linear stress can equilibrate the normal force and the angular momentum. The corresponding 3D element is drawn in Figure 2. The tangential displacement is prescribed by one value for each edge, and the normal component of the normal stress vector is given as linear function on each face. Again, these six degrees of freedom fix the rigid body motions: Displacements fix the tangential displacement and the torsion, while the stresses compensate the normal force, and the two angular moments. The complete basis functions including the internal bubbles will be given in Section 3.

In Figure 3, the degrees of freedom of a quadrilateral element are drawn. The goal is to discretize thin domains with anisotropic elements. Standard primal low order methods lead to bad results as the aspect ratio of the element becomes large. In Section 6 we will analyze the robustness of the corresponding Reissner Mindlin plate element. Now, we motivate the anisotropic element by distinguishing stretching and bending degrees of freedom: The mean value of the horizontal displacement prescribed by the bottom and top edge, and the mean values of the stresses at the vertical edges prescribe a one dimensional stretching deformation. The arising method is a mixed method with continuous stresses and non-continuous displacements. It corresponds to the presented elements, since this is a normal-normal continuous continuous...
1 × 1 stress tensor, and a displacement vector with non-continuous normal components. A bending deformation is prescribed by the vertical deformation along the short edges, the differences of the horizontal deformations, i.e., the rotations, and the differences of the horizontal stresses, i.e., the bending moments. This leads to a stable element for the Timoshenko beam model. A similar dimension reduction works out for thin prismatic and hexahedral elements.

2. Different versions of the Hellinger Reissner formulation

The equations of linear elasticity consist of the constitutive relation

\[ A\sigma = \varepsilon(u), \]

where \( u \) is the unknown displacement vector, \( \varepsilon(u) := \frac{1}{2}[\nabla u + (\nabla u)^T] \) is the strain, \( \sigma \) is the unknown symmetric stress tensor. \( A \) is the compliance tensor, which read as

\[ A\sigma = \frac{1}{2\mu}\sigma^D + \frac{1}{\lambda + \mu}\text{tr}(\sigma)I \]

for an isotropic material with Lamé coefficients \( \mu \) and \( \lambda \). Here, \( \text{tr}(\tau) \) denotes the trace of the tensor \( \tau \), and \( \tau^D \) is the deviator

\[ \tau^D = \tau - \frac{1}{d}\text{tr}(\tau)I. \]

The stress tensor shall be in equilibrium with the prescribed force vector \( f \), i.e.,

\[ \text{div } \sigma = -f, \]

where the div-operator is applied to each row of the tensor. We assume displacement boundary conditions

\[ u = u_D \]

at the part \( \Gamma_D \) of the boundary, and traction boundary conditions

\[ \sigma_n = g \]
at the part $\Gamma_N$ of the boundary. We write $\sigma_n$ for the normal stress vector $\sigma n$ where $n$ is the outer normal vector.

The primal mixed formulation is: find $\sigma \in L^2_{\text{SYM}} := \{ \tau \in L^d_{\text{d} \times \text{d}} : \tau = \tau^T \}$ and $u \in [H^1]^d$ such that $u = u_D$ on $\Gamma_D$ and

(3) \[ \int A \sigma : \tau - \int \varepsilon(u) : \tau = 0 \quad \forall \tau \in L^2_{\text{SYM}}, \]
\[ - \int \varepsilon(v) : \sigma = \int f \cdot v + \int_{\Gamma_N} g \cdot v \quad \forall v \in [H^1_0, D]^d. \]

Since the stresses are $L^2$ variables, they can be eliminated pointwise from the first equation, and one ends up with the primal form: find $u \in [H^1]^d$ such that $u = u_D$ and

(4) \[ \int A^{-1} \varepsilon(u) : \varepsilon(v) = \int f \cdot v + \int_{\Gamma_N} g \cdot v \quad \forall v \in [H^1_0, D]^d. \]

Alternatively, one may integrate by parts the off-diagonal terms, and obtains the dual mixed formulation: find $\sigma \in H(\text{div})_{\text{SYM}} := \{ \tau \in L^2_{\text{d}} : \text{div} \tau \in L^2_{\text{d}} \}$ and $u \in [L^2]^d$ such that $\sigma_n = g$ on $\Gamma_N$ and

(5) \[ \int A \sigma : \tau + \int u \text{div} \tau = \int_{\Gamma_D} u_D \tau_n \quad \forall \tau \in H(\text{div})_{\text{SYM}}, \]
\[ \int v \cdot \text{div} \sigma = - \int f \cdot v \quad \forall v \in [L^2]^d. \]

The spaces imply the necessary continuity of conforming finite element sub-spaces. The primal mixed formulation requires continuous elements for the displacements, but allows discontinuous elements for the stresses. The dual mixed formulation needs elements with continuous normal-vector for the stresses, but the displacements may be discontinuous. Our goal is to end up with elements, where the tangential component of the displacement and the normal component of the normal stress vector are continuous.

The equilibrium equation (2) can be understood in different ways. Posing the equation in $L^2_2$ leads to the bilinear-form

\[ b(\sigma, v) = \int \text{div} \sigma \cdot v, \]

and requires $v \in L^2_2$ as well as $\text{div} \sigma \in L^2_2$. Testing the equilibrium equation with displacements in $H^1$, i.e., posing the equation in $[H^1]^* = H^{-1}$ leads to the form

\[ b(\sigma, v) := \langle \text{div} \sigma, v \rangle_{H^{-1} \times H^1} = \int \sigma : \varepsilon(v). \]

This needs $v \in H^1$, and $\text{div} \sigma \in H^{-1}$. The later one is clearly satisfied by $\sigma \in L^2_2$. 


Our new method tests the equilibrium equation with displacements in $H(\text{curl})$:

$$
\langle \text{div} \sigma, v \rangle_{H(\text{curl})^* \times H(\text{curl})} = \langle f, v \rangle_{H(\text{curl})^* \times H(\text{curl})} \quad \forall v \in H(\text{curl})
$$

The space is

$$
H(\text{curl}) = \{ v \in [L_2]^d : \text{curl} v \in L_2 \},
$$

where the differential operator $\text{curl}$ is defined by

$$
\text{curl} v = \nabla \times v \quad \text{for } d = 3
$$

and

$$
\text{curl} v = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial x} \quad \text{for } d = 2.
$$

This allows right-hand sides of the form

$$
\langle f, v \rangle = \int f_1 \cdot v + \int f_2 \cdot \text{curl} v,
$$

where $f_1$ is a usual volume force density, and $f_2$ is a rotational force density.

**Lemma 1.** The dual space of $H(\text{curl})$ is

$$
H^{-1}(\text{div}) = \{ q \in [H^{-1}]^d : \text{div} q \in H^{-1} \}.
$$

**Proof.** Any function $v \in H(\text{curl})$ can be decomposed as

$$
v = \nabla \varphi + z
$$

with $\varphi \in H^1$ and $z \in [H^1]^d$ and the corresponding bounds in the norms (see [19], Theorem 3.4). The dual norm is

$$
\| q \|_{H(\text{curl})^*} = \sup_{v \in H(\text{curl})} \frac{\langle q, v \rangle}{\| v \|_{H(\text{curl})}} \approx \sup_{v \in H(\text{curl})} \inf_{\varphi = \nabla \varphi + z} \| \varphi \|_{H^1} + \| z \|_H = \sup_{\varphi, z} \frac{\langle q, \nabla \varphi + z \rangle}{\| \varphi \|_{H^1} + \| z \|_H} \approx \sup_{\varphi \in H^1} \frac{\langle q, \nabla \varphi \rangle}{\| \varphi \|_{H^1}} + \sup_{z \in [H^1]^d} \frac{\langle q, z \rangle}{\| z \|_H} = \sup_{\varphi \in H^1} \frac{\langle \text{div} q, \varphi \rangle}{\| \varphi \|_{H^1}} + \sup_{z \in [H^1]^d} \frac{\langle q, z \rangle}{\| z \|_H} = \| \text{div} q \|_{H^{-1}} + \| q \|_{H^{-1}}
$$

$\square$
The proper function space $\Sigma$ for the stresses is

$$\Sigma = \{ \sigma \in L^2_{\text{SYM}} : \text{div} \sigma \in H^{-1}(\text{div}) \}$$

The constraint $\text{div} \sigma \in H^{-1}$ is no new requirement, the only additional constraint is $\text{div} (\text{div} \sigma) \in H^{-1}$. Thus, the space can be written also as

$$\Sigma = \{ \sigma \in L^2_{\text{SYM}} : \text{div} \text{div} \sigma \in H^{-1} \}$$

Throughout the following, let $\mathcal{T}_h = \{ T \}$ be a shape-regular triangulation of $\Omega$, consisting of triangular or tetrahedral elements $T$ in two or three space dimensions, respectively. Let $\mathcal{F}$ denote the set of all element interfaces (facets), i.e. edges in two or faces in three space dimensions, respectively. In 3d, we also define $\mathcal{E}$ as the set of all element edges.

The following theorem provides that normal-normal continuous finite elements are proper for the space $\Sigma$:

**Theorem 2.** Assume that $\sigma$ is a piece-wise smooth symmetric tensor field such that $\sigma_{n\tau} \in H^{1/2}(\partial T)$ and $n^T \sigma n$ is continuous across interfaces. Then there holds $\text{div} \sigma \in H(\text{curl})^*$. 

**Proof.** Let $v$ be a smooth test function. The divergence operator $\text{div} : H(\text{div div}) \rightarrow H(\text{curl})^*$ is then defined as

$$\langle \text{div} \sigma, v \rangle = - \int_\Omega \sigma : \nabla v = \sum_{T \in \mathcal{T}} \left\{ \int_T \text{div} \sigma \cdot v - \int_{\partial T} \sigma_{n} \cdot v \right\}.$$

We further investigate the boundary terms. Using the continuity of $\sigma_{nn}$ and $u_\tau$ across element interfaces, we may reorder the surface integrals facet by facet. We can write

$$\sum_{T \in \mathcal{T}} \int_{\partial T} \sigma_{n} \cdot v = \sum_{T \in \mathcal{T}} \int_{\partial T} \sigma_{nn} v_n + \sigma_{n\tau} v_\tau$$

$$= \sum_{F \in \mathcal{F}} \int_F \sigma_{nn} [v_n] + \sum_{T \in \mathcal{T}} \int_{\partial T} \sigma_{n\tau} v_\tau$$

Using this relation, we obtain

$$\langle \text{div} \sigma, v \rangle = \sum_T \left\{ \int_T \text{div} \sigma \cdot v - \int_{\partial T} \sigma_{n\tau} v_\tau \right\} + \sum_F \int_F [\sigma_{nn}] v_n$$

$$\leq \sum_T \| \text{div} \sigma \|_{L^2(T)} ||v||_{L^2(T)} + ||\sigma_{n\tau}||_{H^{1/2}(\partial T)} ||v_\tau||_{H^{-1/2}(\partial T)}$$

$$\leq C(\sigma) ||v||_{H(\text{curl})}$$
By density, the continuous functional can be extended to the whole space $H(\text{curl})$:

$$\langle \text{div } \sigma, v \rangle = \sum_T \left\{ \int_T \text{div } \sigma \cdot v - \int_{\partial T} \sigma_{n\tau} v_\tau \right\}$$

\[\square\]

**Remark 3.** The condition $\sigma_{n\tau} \in H^{1/2}$ is a continuity constraint at vertices in 2d and at edges in 3d. If we slightly weaken the regularity, the constraint disappears, and much simpler elements can be used. The analysis in discrete norms in the next section will rectify this violation of conformity.

Let us now include boundary conditions into the problem formulation. We assume that $u$ is given on a non-trivial part $\Gamma_D \subseteq \Gamma$ of the boundary, and surface tractions $\sigma_n$ are prescribed on $\Gamma_N = \Gamma \setminus \Gamma_D$. For simplicity, let all data be homogenous,

$$u = 0 \text{ on } \Gamma_D, \quad \sigma_n = 0 \text{ on } \Gamma_N.$$ 

Boundary conditions for $u_\tau, \sigma_{nn}$ turn out to be essential, conditions on $u_n, \sigma_{n\tau}$ are natural and can be included into the variational equations. More precisely, we search for $(\sigma, u) \in \Sigma_0 \times V_0$ where

$$\Sigma_0 = \{ \sigma \in H(\text{div div}) : \sigma_{nn} = 0 \text{ on } \Gamma_N \}$$

$$V_0 = \{ v \in H(\text{curl}) : v_\tau = 0 \text{ on } \Gamma_D \}$$

If the solution of the elasticity problem is piecewise smooth, the duality pairing $\langle \text{div } \sigma, v \rangle$ can be rewritten by means of domain and boundary integrals. Now, we formulate the problem by means of these integrals. For smooth solutions, this is an equivalent formulation of the elasticity problem.

**Theorem 4.** Let the solutions be piece-wise smooth. Then the elasticity problem (1),(2) is equivalent to the following mixed problem: Find $\sigma \in \Sigma_0$ and $u \in V_0$ such that

$$\int A \sigma \cdot \tau + \sum_T \left\{ \int_T \text{div } \tau \cdot u - \int_{\partial T} \tau_{n\tau} u_\tau \right\} = 0 \quad \forall \tau \in \Sigma_0$$

$$\sum_T \left\{ \int_T \text{div } \sigma \cdot v - \int_{\partial T} \sigma_{n\tau} v_\tau \right\} = -\int f \cdot v \quad \forall v \in V_0$$

**Proof.** Again, we reorder the surface integrals on element interfaces $\partial T$ in a face-wise mode. Then the second line turns out to be the equilibrium equation, plus tangential continuity of the normal stress vector:

$$\sum_T \int_T (\text{div } \sigma + f) v + \sum_{F} \int_F [\sigma_{n\tau}] v_\tau = 0 \quad \forall v$$
Since the space requires continuity of $\sigma_{nn}$, the normal stress vector is continuous.

Element-wise integration by parts in the first line gives

$$\sum_T \int_T (A\sigma - \varepsilon(u)) : \tau + \sum_F \int_F \tau_{nn}[u_n] = 0 \quad \forall \tau$$

This is the constitutive relation, plus normal-continuity of the displacement. Tangential continuity of the displacement is implied by the space $H(\text{curl})$.

3. Finite element error estimates

The displacement space $H(\text{curl})$ and the stress space $H(\text{div\ div})$ imply the necessary continuity of finite element spaces: to be conforming, the tangential component of the displacement as well as the normal-normal tensor components need to be continuous.

For $k \geq 1$, we define the finite element spaces

$$\Sigma_h := \{ \tau_h \in L_2^{\text{SYM}} : \tau_h|_T \in P^k, \tau_{nn} \in P^{\max(1,k-1)} \text{ continuous}, \tau_{nn} = 0 \text{ on } \Gamma_N \}$$

$$V_h := \{ v_h \in (L_2)^d : v_h|_T \in P^k, v_\tau \text{ continuous}, v_\tau = 0 \text{ on } \Gamma_D \}.$$  

The space $V_h$ is the second family of Nédélec. Up to our knowledge, the space $\Sigma_h$ is a new finite element space.

In the continuous setting, both $v = H(\text{curl})$ and $\Sigma = H(\text{div\ div})$ are equipped with their natural norms. For the finite element analysis, we choose different norms. We define the discrete, mesh dependent norm for the displacements

$$\|v\|_{V_h}^2 := \sum_T \|\varepsilon(v)\|_{L_2(T)}^2 + \sum_{F \in \mathcal{F}} h_F^{-1} \|[v_n]\|_{L_2(F)}^2.$$  

Note that, using the piecewise strain tensor instead of gradients, we will not require Korn’s inequality [18, 22]. This will be useful when treating anisotropic geometries or elements, for which the constant in Korn’s inequality deteriorates. For the stress space, we use the $L^2$ norm

$$\|\sigma\|_{\Sigma_h} := \|\sigma\|_{L_2}.$$  

There the local mesh size $h_F$ is the average height perpendicular to $F$ of the two adjacent elements $\Delta_F = \{ T_F^+, T_F^- \}$. Then there holds

$$h_F \lesssim \frac{|\Delta_F|}{|F|}.$$  

We can pose the finite dimensional problem
Find \( \sigma \in \Sigma_h, u \in V_h \) such that

\[
\begin{align*}
\mathbf{a}(\sigma, \tau) + b(\tau, u) &= 0 \quad \forall \tau \in \Sigma_h \\
b(\sigma, v) &= -(f, v) \quad \forall v \in V_h.
\end{align*}
\]

The bilinear forms \( a : \Sigma \times \Sigma \rightarrow \mathbb{R} \) is defined via (6). In the homogenous, isotropic case, the compliance tensor \( A \) is determined by the Lamé constants \( \lambda \) and \( \mu \) only. In this case \( a \) is given by (7). Here \( \text{tr}(\tau) \) denotes the trace of the tensor \( \tau \), and \( \tau^D = \tau - \frac{1}{d} \text{tr}(\tau) I \).

\[
\begin{align*}
(6) \quad \mathbf{a}(\sigma, \tau) &= \int_{\Omega} A\sigma : \tau \, dx \\
(7) &= \int_{\Omega} \frac{1}{2\mu} \sigma^D : \tau^D + \frac{1}{\lambda + \mu} \text{tr}(\sigma) \text{tr}(\tau) \, dx.
\end{align*}
\]

The second bilinear form \( b : \Sigma \times V \rightarrow \mathbb{R} \) is generated by the divergence operator. For piecewise smooth functions \( \tau, v \) it can be evaluated by means of integrals over elements \( T \) and element boundaries \( \partial T \). Using integration by parts, we can find two different formulations defining \( b \):

\[
\begin{align*}
(8) \quad b(\tau, v) &= \langle \text{div} \tau, v \rangle \\
(9) &= \sum_T \int_T \text{div} \tau \cdot v \, dx - \int_{\partial T} \tau_n \cdot v_r \, ds \\
(10) &= \sum_T -\int_T \tau : \varepsilon(v) \, dx + \int_{\partial T} \tau_m v_n \, ds.
\end{align*}
\]

Combining these two forms into one, we also define the compound bilinear form \( \mathcal{B} : (\Sigma \times V) \times (\Sigma \times V) \rightarrow \mathbb{R} \) by

\[
\mathcal{B}(\sigma, u; \tau, v) = a(\sigma, \tau) + b(\sigma, v) + b(\tau, u).
\]

### 3.1. Discrete stability

In the continuous setting, we have continuity and inf-sup stability of the bilinear form \( \mathcal{B} \) on \( \Sigma_0 \times V_0 \). In this subsection, we are concerned with an analogous condition for the discrete problem. It is sufficient to show that [14]

1. Continuity of \( a, b \)

\[
\begin{align*}
a(\sigma, \tau) &\leq \tilde{a}_2 \|\sigma\|_{\Sigma_h} \|\tau\|_{\Sigma_h} \quad \forall \sigma, \tau \in \Sigma_h \\
b(\sigma, v) &\leq \tilde{b}_2 \|\sigma\|_{\Sigma_h} \|v\|_{V_h} \quad \forall \sigma \in \Sigma_h, v \in V_h
\end{align*}
\]

2. Coercivity on the kernel

\[
a(\sigma, \sigma) \geq \tilde{\alpha}_1 \|\sigma\|_{\Sigma_h}^2 \quad \forall \sigma \in \text{Ker} \, \mathcal{B}_h
\]
(3) inf-sup stability

$$\inf_{v \in V_h} \sup_{\sigma \in \Sigma_h} \frac{b(\sigma, v)}{\|\sigma\|_{\Sigma_h} \|v\|_{V_h}} \geq \tilde{\beta}_1$$

We will see that both conditions are satisfied, but the stability constant $\tilde{\alpha}_1$ depends on material parameters.

For isotropic material, the compliance tensor $A$ is determined by the Lamé constants $\lambda$ and $\mu$. As $\lambda$ approaches infinity, the material becomes incompressible. Then $a$ is not uniformly coercive on the whole space $\Sigma_0$. According to the second condition above, we only need coercivity on the kernel of $B$. In the continuous case, $\text{Ker} B$ is the set of exactly divergence free functions. One can show that, for divergence free functions $\sigma$, there holds

$$a(\sigma, \sigma) \geq \alpha_1 \|\sigma\|_{\Sigma}^2$$

where the constant $\alpha_1 > 0$ does not depend on $\lambda$ [14].

The bilinear form $a$ is not uniformly coercive on the discrete kernel

$$\text{Ker} B_h = \{\sigma \in \Sigma_h : b(\sigma, v) = 0 \ \forall v \in V_h\}.$$  

To stabilize the problem, we add a further term to the bilinear form $a$ element by element. We define the stabilized form

$$a^s(\sigma, \tau) = (A\sigma, \tau)_{0,\Omega} + \sum_T h^2(\text{div} \sigma, \text{div} \tau)_{0,T}.$$  

The solution $\sigma$ satisfies $-\text{div} \sigma = f$. Therefore, to preserve consistency, the term $-h^2(f, \text{div} \tau)_{0,T}$ has to be added element-wise to the right hand side. We obtain the stabilized problem

Find $\sigma \in \Sigma_h, u \in V_h$ such that

$$a^s(\sigma, \tau) + \langle \text{div} \tau, u \rangle = -\sum_T h^2(f, \text{div} \tau)_{0,T} \ \forall \tau \in \Sigma_h$$

$$\langle \text{div} \sigma, v \rangle = -(f, v) \ \forall v \in V_h.$$  

We will see that the stabilized form

$$B^s(\sigma, u; \tau, v) = a^s(\sigma, \tau) + b(\sigma, v) + b(\tau, u)$$

is inf-sup stable by proving an inf-sup condition for $b$ and coercivity of $a^s$ on the discrete kernel, where the stability constants do not depend on $\lambda$.

In the following, we use the $H(\text{curl})$ and the $H(\text{div div})$ conforming transformations from the reference element $\hat{T}$ to the physical element $T$, which are described more closely in Subsection 4.3. They are given by

$$v(x) = F_T^{-T} \hat{v}(\hat{x})$$

$$\sigma(x) = \frac{1}{J_T^2} (F_T \hat{\sigma}(\hat{x}) F_T^T)$$
Here $F_T$ is the Jacobian of the transformation $\Phi_T : \hat{T} \to T$, and $J_T = \det F_T$ denotes the Jacobi determinant. Note that the tangential components of vector fields and the normal-normal components of tensor-valued functions are not changed by these transformations.

When treating anisotropic geometries, it will be useful to use different norms in the discrete spaces. For the displacement space, instead of taking the piecewise defined gradient, we use the strain tensor $\varepsilon(u)$ on each element. As we will see, all stability estimates are independent of the shape of the physical element.

**Lemma 5.** Assume $T$ to be a shape-regular triangularization of the domain $\Omega$. Then the norm for the displacement space $\| \cdot \|_{V_h}$ is equivalent to the broken $H^1$ semi norm

\[
\|v\|_{V_h}^2 \simeq \|v\|_{\tilde{V}_h}^2 := \sum_T \|\varepsilon(v)\|_{L^2(T)}^2 + \sum_{F \in \mathcal{F}} h_F^{-1} \|P_1[v_n]\|_{L^2(F)}^2.
\]

Moreover, for the discrete stress space we have

\[
\|\sigma\|_{\Sigma_h}^2 \simeq \|\sigma\|_{\tilde{\Sigma}_h}^2 := \sum_T \|\sigma\|_{L^2(T)}^2 + \sum_{F \in \mathcal{F}} h_F \|\sigma_{nn}\|_{L^2(F)}^2.
\]

**Proof.** We prove the equivalence of the norms $\| \cdot \|_{V_h}$ and $\| \cdot \|_{\tilde{V}_h}$ first. This equivalence also holds true in the space of $H(\text{curl})$-conforming functions, which are piecewise smooth:

\[
V_T := \{v \in V_0 : v|_T \in H^1(T) \ \forall T \in \mathcal{T}\}.
\]

Clearly, there holds $\|v\|_{\tilde{V}_h} \leq \|v\|_{V_h}$ We will now prove the opposite bound, namely $\|v\|_{V_h} \leq \|v\|_{\tilde{V}_h}$. Let $\Pi_T$ denote the element-wise projection onto the space of piecewise rigid body motions $\text{RM}(T) = \{a + bx : a \in \mathbb{R}^d, b \in \mathbb{R}^{skw}\}$. This projection is defined via

\[
\int_T v - \Pi_T v \, dx = \int_T \text{curl}(v - \Pi_T v) \, dx = 0.
\]

As $\text{RM}(T)$ is a subspace of $P^1(T)$, there holds for any facet $F \in \mathcal{F}$

\[
\|(I-P^1)[v]_n\|_{L^2(F)} = \|(I-P^1[v - \Pi_T(v)]_n\|_{L^2(F)} \leq \|[v - \Pi_T(v)]_n\|_{L^2(F)}.
\]

Following [13], there holds

\[
\|[v - \Pi_T(v)]_n\|_{L^2(F)} \leq \sum_{T \in \Delta_F} h_F \|\varepsilon(v)\|_{L^2(T)}^2.
\]
Combining these estimates, we obtain
\[
\sum_F h_F^{-1} \|v_n\|_{L^2(F)}^2 = \sum_F h_F^{-1} \|P^1[v_n]\|_{L^2(F)}^2 + h_F^{-1} \|(I-P^1)[v_n]\|_{L^2(F)}^2
\]
\[
\lesssim \sum_F h_F^{-1} \|P^1[v_n]\|_{L^2(F)}^2 + \sum_T \|\varepsilon(v)\|_{L^2(T)}^2 = \|v\|_{\tilde{V}_h}
\]

This relation proves the first equivalence.

For the stress space \(\Sigma_h\), we transform \(\sigma\) element-wise to the reference element \(\hat{T}\). There we use the equivalence of norms on a finite-dimensional space, and transform back. This gives
\[
\|\sigma\|_{L^2(\Omega)}^2 = \sum_T \int_T \sigma : \sigma \, dx = \sum_T \int_{\hat{T}} \frac{1}{J_T^2} |F_T \hat{\sigma} F_T^T|^2 |T| \, d\hat{x}
\]
\[
\approx \sum_T \int_{\partial T} (\hat{n}^T \hat{\sigma} \hat{n})^2 |T| \, d\hat{s} + \sum_T \int_{\hat{T}} \frac{1}{J_T^2} |F_T \hat{\sigma} F_T^T|^2 |T| \, d\hat{x}
\]
\[
= \sum_T \int_{\partial T} (n^T \sigma n)^2 |T| \frac{|T|}{|F|} \, ds + \|\sigma\|_{L^2(\Omega)}^2 = \|\sigma\|_{\tilde{\Sigma}_h}^2
\]

\(\square\)

Lemma 6. The bilinear forms \(a^\ast : \Sigma_h \times \Sigma_h \to \mathbb{R}\), \(b : \Sigma_h \times V_h \to \mathbb{R}\) are continuous.

Proof. To show continuity of \(a^\ast\) in the discrete norms, we need the inverse inequality
\[
\|\text{div} \, \sigma\|_{L^2(T)} \lesssim h^{-1} \|\sigma\|_{L^2(T)} \quad \forall \sigma \in \Sigma_h.
\]

Then we can estimate
\[
a^\ast(\sigma, \tau) = \int_{\Omega} \frac{1}{2\mu} \sigma^D : \tau^D + \frac{1}{\lambda + \mu} \text{tr}(\sigma) \text{tr}(\tau) \, dx + \sum_T h^2(\text{div} \, \sigma, \text{div} \, \tau)_{L^2(T)}
\]
\[
\lesssim \left( \frac{1}{2\mu} + \frac{1}{\lambda + \mu} \right) \|\sigma\|_{L^2} \|\tau\|_{L^2} + \sum_T \|\sigma\|_{L^2(T)} \|\tau\|_{L^2(T)}
\]
\[
= \tilde{\alpha}_2 \|\sigma\|_{\Sigma_h} \|\tau\|_{\Sigma_h}.
\]
For the second bilinear form, we use the norm equivalences from above and see

\[
b(\sigma, v) = \sum_T - \int_T \sigma : \varepsilon(v) \, dx + \int_{\partial T} \sigma_{nn} v_n \, ds \leq \sum_T \|\sigma\|_{L^2(T)} \|\varepsilon(v)\|_{L^2(T)} + \sum_F h_F^{1/2} \|\sigma_{nn}\|_{L^2(F)} h^{-1/2} \|v_n\|_{L^2(F)} \leq \|\sigma\|_{\Sigma_h} \|v\|_{V_h} \leq \tilde{\beta}_2 \|\sigma\|_{\Sigma_h} \|v\|_{V_h}.
\]

□

**Lemma 7.** The bilinear form \(b : \Sigma_h \times V_h \to \mathbb{R}\) is inf-sup stable, there exists a positive constant \(\tilde{\beta}_1 > 0, \tilde{\beta}_1\) independent of \(h\) such that

\[
(11) \inf_{v \in V_h} \sup_{\sigma \in \Sigma_h} \frac{b(\sigma, v)}{\|\sigma\|_{\Sigma_h} \|v\|_{V_h}} \geq \tilde{\beta}_1.
\]

**Proof.** The finite element space \(\Sigma_h\) can be decomposed into two parts

\[\Sigma_h = \Sigma^f_h + \Sigma^b_h.\]

The first space, \(\Sigma^f_h\), is associated to the set of faces \(\mathcal{F}\). For each face \(F \in \mathcal{F}\), there exists a family of functions \((\sigma^F_j)_{0 \leq j \leq k}\) such that

\[|\sigma^F_j|_F = 0 \quad \forall F \in \mathcal{F}, \bar{F} \neq F; \]

\[
\{\sigma_{nn}^F : \sigma^F \in \Sigma^f_h\} = P^k(F)
\]

The second space, \(\Sigma^b_h\), consists of element bubble functions, which satisfy

\[\sigma_{nn}^b |_F = 0 \quad \forall F \in \mathcal{F}.\]

To construct these spaces, we need tensor-valued basis functions \(S^f_i\) corresponding to the facets of the element. These basis functions span \(P^0\) on a single element \(T\), and the normal-normal component vanishes on all facets but one, where it is constant. We will now define the basis on the reference element \(\hat{T}\). Therefore, let \((\lambda_i)\) denote the barycentric coordinates of the vertices. In 2d, we propose

\[\hat{S}^f_i = \text{sym}[\nabla \lambda^\perp_{i+1} \otimes \nabla \lambda^\perp_{i+2}], i = 1 \ldots 3.\]

Calculating them explicitly yields

\[
\hat{S}^f_1 = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}, \hat{S}^f_2 = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}, \hat{S}^f_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
In 3d, we need four face-based functions $\hat{S}_f^i$ and two additional element bubbles $\hat{S}_t^i$ to span the six-dimensional space of constant symmetric $3 \times 3$ tensor fields. We set them via

$$\hat{S}_f^i = \text{sym}[(\nabla \lambda_{i+1} \times \nabla \lambda_{i+2}) \otimes (\nabla \lambda_{i+2} \times \nabla \lambda_{i+3})]$$

$$\hat{S}_t^1 = \text{sym}[(\nabla \lambda_1 \times \nabla \lambda_2) \otimes (\nabla \lambda_3 \times \nabla \lambda_4)]$$

$$\hat{S}_t^2 = \text{sym}[(\nabla \lambda_1 \times \nabla \lambda_3) \otimes (\nabla \lambda_2 \times \nabla \lambda_4)].$$

Explicitely, they read as

$$\hat{S}_f^1 = \begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{S}_f^2 = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad \hat{S}_f^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\hat{S}_f^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \hat{S}_t^1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad \hat{S}_t^2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$  

They can be transformed to the physical element $T$ using the transformation given in Subsection 4.3. Using these transformations we can define a piecewise constant tensor field $S^F$ for any facet $F$ such that $S^F_{nn}|_F = \delta_{FF} \quad \forall F.$

Let $v \in V_h$ be given. In order to prove the Lemma, we want to find some $\sigma \in \Sigma_h$ satisfying

$$b(\sigma, v) = \sum_T \int_T \text{div} \sigma \cdot v \, dx - \int_{\partial T} \sigma_{nT} \cdot v_T \, ds \geq \tilde{\beta}_1 \|\sigma\|_0 \|v\|_{V_h},$$

$$\|\sigma\|_{\Sigma_h} \leq c \|v\|_{V_h}.$$  

Therefore we decompose the stress function

$$\sigma = \alpha \sigma^b + \beta \sigma^f,$$

where $\alpha, \beta \in \mathbb{R}$ are to be specified later. Let $\sigma^f$ be such that $\sigma_{nn}^f|_F = h_F^{-1}[v_n]|_F \quad \forall F \in \mathcal{F}.$

This is possible as $[v_n]$ is a polynomial of degree $k$ on each face, and therefore we can simply set

$$\sigma^f := \sum_F h_F^{-1}[v_n]|_F S^F.$$  

Note that, when using $\| \cdot \|_{\tilde{V}_h}$, we can choose $\sigma^f$ of order one. The approximation properties of $\Sigma_h \times V_h$ are still of optimal order when using facet bubbles of order $k - 1.$
Next, we determine the “bubble part” $\sigma^b$ on each element $T$. In 2d, the space of $P^1$ bubble functions on the reference element is spanned by
\[
\hat{\Sigma}^b := \{ \hat{S}_i^F \lambda_i, i = 1, 2, 3 \}.
\]
In 3d, we have to add some more functions
\[
\hat{\Sigma}^b \{ \hat{S}_i^F \lambda_i, i = 1 \ldots 4 \} \cup \{ \hat{S}_j^T P^1(\hat{T}), i = 1, 2 \}.
\]
Note that the space of constant symmetric tensor fields on the reference element is spanned in 2d by $\{ \hat{S}_i^F \}$, in 3d by $\{ \hat{S}_i^F, \hat{S}_j^T \}$ respectively.

For simplicity, we will do the following calculations for the 2d case. To treat three space dimensions, we would need to add the additional bubble functions. Choose the “bubble part” $\sigma^b$ such that, on each triangle resp. tetrahedron $T$, $\hat{\sigma}^b = J_T^2 (F_T^{-1} \sigma^b F_T^{-T})$ is given by
\[
\hat{\sigma}^b = J_T^2 \sum_i (\hat{\epsilon}(\hat{v}) : F_T^{-T} \hat{S}_i^F F_T^{-1}) F_T^{-T} \hat{S}_i^F F_T^{-1} \lambda_i.
\]
To achieve a bound for the $L^2$-norm of $\sigma$, we estimate the two parts $\sigma^b, \sigma^f$ separately by $\|v\|_{V_h}$. We will see in both cases, that the constant arising is independent of the shape or the physical element. This will prove useful when treating anisotropic geometries.

First, we treat $\sigma^f$. Similar to the proof in Lemma 5, we obtain
\[
\|\sigma^f\|_{L^2(\Omega)}^2 \simeq \sum_F h_F \|\sigma^f_{nn}\|_{L^2(F)}^2 = \sum_F h_F^{-1} \|[v_n]\|_{L^2(F)}^2.
\]
Here we used that $\sigma^f$ consists of edge bubble functions only, therefore the facet-contributions of $\sigma_{nn}$ only form a norm on the discrete space.

For the bubble part $\sigma^b$, there holds
\[
\|\sigma^b\|_{L^2(\Omega)}^2 = \alpha \sum_T \int_{\hat{T}} \frac{1}{J_T^2} |F_T^{-T} \hat{\sigma}^b F_T|^2 J_T d\hat{x}
\]
\[
\simeq \sum_T \int_{\hat{T}} \sum_i |(\hat{\epsilon}(\hat{v}) : F_T^{-T} \hat{S}_i^F F_T^{-1}) \hat{S}_i^F \lambda_i|^2 J_T d\hat{x}
\]
\[
= \sum_T \int_{\hat{T}} \sum_i |(F_T^{-1} \hat{\epsilon}(\hat{v}) F_T^{-T} : \hat{S}_i^F) \hat{S}_i^F \lambda_i|^2 J_T d\hat{x}
\]
\[
\leq \sum_T \int_{\hat{T}} |F_T^{-1} \hat{\epsilon}(\hat{v}) F_T^{-T}|^2 J_T d\hat{x}
\]
\[
= \sum_T \|\epsilon(v)\|_{L^2(T)}^2.
\]
Combining the two inequalities, we obtain

$$\|\sigma\|_{L^2(\Omega)}^2 \leq \alpha c_1 \sum_T \|\varepsilon(v)\|_{L^2(T)}^2 + \beta c_2 \sum_F h_F^{-1} \|\nabla v_n\|_{L^2(F)}^2$$

We will need the following bound for the bubble part $\sigma^b$ on each element

$$\int_T \sigma^b : \varepsilon(v) \, dx \geq c_3 \|\varepsilon(v)\|_{L^2(T)}^2.$$

Again, we prove this by transformation to the reference element $\hat{T}$. We obtain

$$\int_T \sigma^b : \varepsilon(v) \, dx = \int_T \frac{1}{J_T^2} (F_T \hat{\sigma}(\hat{x}) F_T^T) : F_T^{-1} \hat{\varepsilon}(\hat{v})(\hat{x}) F_T^{-1} J_T d\hat{x}$$

$$= \int_T \frac{1}{J_T^2} \hat{\sigma} : \hat{\varepsilon}(\hat{v}) J_T d\hat{x}$$

$$= \int_T \sum_i (\hat{\varepsilon}(\hat{v}) : \hat{S}_i^F F_T^{-1} \hat{S}_i^F F_T^{-1} \lambda_i : \hat{\varepsilon}(\hat{v}) J_T d\hat{x})$$

$$= \int_T \sum_i (F_T^{-1} \hat{\varepsilon}(\hat{v}) F_T^{-1} : \hat{S}_i^F)^2 \lambda_i J_T d\hat{x}$$

$$\leq \int_T |F_T^{-1} \hat{\varepsilon}(\hat{v}) F_T^{-1}|^2 J_T d\hat{x}$$

$$= \int_T |\varepsilon(v)|^2 \, dx = \|\varepsilon(v)\|_{L^2(T)}^2$$

where we used that the functions $(S_i)_i$ form a basis for $P^{2\times2,SYM}_0$. 

For \( \sigma \) defined as above we apply Young’s inequality and obtain

\[
\begin{align*}
\mathbf{b}(\sigma,v) &= \sum_T \int_T \varepsilon(v) : \sigma \, dx - \sum_F \int_F \sigma_{nn}[v_n] \, ds \\
&= \sum_T \int_T [\alpha \sigma^b : \varepsilon(v) + \beta \sigma^f : \varepsilon(v)] \, dx - \beta \sum_F \int_F \sigma^f_{nn}[v_n] ds \\
&\geq \sum_T \alpha c_3 \|\varepsilon(v)\|_{L^2(T)}^2 - \beta \|\sigma^f\|_{L^2(T)} \|\varepsilon(v)\|_{L^2(T)} + \beta h_F^{-1} \|\sigma_n\|_{L^2(F)}^2 \\
&\geq \sum_T \alpha c_3 \|\varepsilon(v)\|_{L^2(T)}^2 - \beta c_2 h_F^{-1/2} \|\sigma_n\|_{L^2(\partial T)} \|\varepsilon(v)\|_{L^2(T)} + \\
&\quad \beta \sum_F h_F^{-1} \|\sigma_n\|_{L^2(F)}^2 \\
&\geq \sum_T \alpha c_3 \|\varepsilon(v)\|_{L^2(T)}^2 - \frac{\beta c_2 \varepsilon^2}{2} \sum_T \|\varepsilon(v)\|_{L^2(T)}^2 \\
&\quad - \frac{\beta c_2}{2 \varepsilon^2} \sum_F h_F^{-1} \|\sigma_n\|_{L^2(F)}^2 + \beta \sum_F h_F^{-1} \|\sigma_n\|_{L^2(F)}^2 \\
&\geq \frac{1}{2} \sum_T \|\varepsilon(v)\|_{L^2(T)}^2 + \frac{1}{2} \sum_F \|\sigma_n\|_{L^2(F)}^2 \\
&\geq \frac{1}{2 c(\alpha, \beta)} \|\varepsilon\|_{V_h} \|\sigma\|_{\Sigma_h}.
\end{align*}
\]

Choosing \( \beta = 1, \alpha = (1 + c_2^2)/(2c_3), \varepsilon^2 = 2c_2 \) we get

\[
\mathbf{b}(\sigma,v) \geq \frac{1}{2} \sum_T \|\varepsilon(v)\|_{L^2(T)}^2 + \frac{1}{2} \sum_F \|\sigma_n\|_{L^2(F)}^2 \\
\geq \frac{1}{2 c(\alpha, \beta)} \|\varepsilon\|_{V_h} \|\sigma\|_{\Sigma_h}.
\]

To show coercivity of \( a^* \), we need the following lemma. We assume that \( \Gamma_N \) is a non-trivial part of the boundary \( \Gamma = \partial \Omega \), where Neumann boundary conditions are prescribed. The Dirichlet problem, i. e. \( \Gamma_N = \emptyset \), is treated separately.

**Lemma 8.** For any \( z \in L_2 \) there exists some \( p \in [H^1]^d, p_r|_F \in P^1 \) for all faces \( F \in \mathcal{F} \), satisfying

\[
\begin{align*}
\text{div} \, p &= z \quad \text{in } \Omega \\
p &= 0 \quad \text{on } \Gamma \backslash \Gamma_N.
\end{align*}
\]

**Proof.** From Stokes theory it is well known that there exists a constant \( \gamma > 0 \) such that

\[
\inf_{z \in L_2} \sup_{p \in [H^1]^d} \frac{(\text{div} \, p, z)_{L^2}}{\|p\|_{H^1} \|z\|_{L^2}} \geq \gamma.
\]
This implies the existence of \( p_1 \in [H^1]^d \) such that
\[
\text{div} \, p_1 = z, \quad p = 0 \text{ on } \Gamma \backslash \Gamma_N.
\]
We are now looking for a polynomial approximation to \( p_1 \). Therefore, we consider the \( H^1 \) conforming finite element space \( W_h \) proposed by [19]. The degrees of freedom of a function \( w \in W_h \) are the vertex values and the mean values of the normal components of \( w \) on the element faces. Restricted to a triangle \( T \) in 2D, \( w \) can be written as
\[
w|_T = \sum_{i=1}^{3} w_i \lambda_i + \sum_{j=1}^{3} \bar{w}_j \lambda_{j+1} \lambda_{j+2} n_j
\]
whereas on a tetrahedron \( T \) in 3D \( w \) takes the form
\[
w|_T = \sum_{i=1}^{4} w_i \lambda_i + \sum_{j=1}^{4} \bar{w}_j \lambda_{j+1} \lambda_{j+2} \lambda_{j+3} n_j.
\]
Note that the tangential component of \( w \in W_h \) still is in \( P^1 \).

From the Stokes problem it is known that there exists a Fortin operator \( \Pi_h : [H^1]^d \to W_h \) such that
\[
\int_F (w - \Pi_h w) \cdot n \, dx = 0 \quad \forall F \in \mathcal{F}
\]
\[
\|\Pi_h w\|_1 \leq c\|w\|_1
\]
Using this operator, we define
\[
p_{1,h} = \Pi_h p_1
\]
which satisfies, due to its construction
\[
\int_T \text{div} \, p_{1,h} \, dx = \int_T \text{div} \, p_1 \, dx = \int_T \text{tr}(\sigma) \, dx.
\]
Next, we choose \( p_2 \) by solving the following problem locally on each triangle resp. tetrahedron \( T \).

Find \( p_2 \in H^1 \), \( p_2|_{\partial T} = 0 \) such that
\[
\text{div} \, p_2 = z - \text{div} \, p_{1,h}.
\]
These local solutions exist, as the right hand side \( z - \text{div} \, p_{1,h} \) has zero mean value. Further \( p_2 \) is in \([H^1]^d\) globally, as it vanishes on element interfaces. Setting \( p = p_{1,h} + p_2 \), we obtain
\[
\text{div} \, p = \text{div} \, p_{1,h} - \text{div} \, p_2 = \text{tr}(\sigma),
\]
i.e. the lemma is proven. \( \square \)

Using the result above, we can prove stability for the bilinear form \( a^s \) in the finite element spaces.
Lemma 9. The bilinear form $a^s : \Sigma_h \times \Sigma_h \rightarrow \mathbb{R}$ is coercive, there exists a constant $\tilde{\alpha}_1$ independent of the Lamé constant $\lambda$ such that

$$a^s(\sigma, \sigma) \geq \tilde{\alpha}_1 \|\sigma\|_{L_h}^2 \quad \forall \sigma \in \text{Ker } B_h$$

where $\text{Ker } B_h$ is the discrete kernel

$$\text{Ker } B_h = \{ \sigma \in \Sigma_h : b(\sigma, v) = 0 \ \forall v \in V_h \}. $$

Proof. For this proof we assume that the boundary part, where $\sigma_n$ is given, is non-trivial, $|\Gamma_N| \neq 0$. For any $\tau \in \Sigma$ there holds

$$\|\tau\|_{L^2}^2 = \|\tau^D\|_{L^2}^2 + \frac{1}{d} \|\text{tr}(\tau)\|_{L^2}^2$$

and further

$$\|\tau\|_{L^2}^2 \leq 2\mu a(\tau, \tau) + \|\text{tr}(\tau)\|_{L^2}^2.$$ 

Therefore, it is sufficient to show that for $\sigma \in \text{Ker } B_h$

$$a(\sigma, \sigma) \geq c \|\text{tr}(\sigma)\|_{L^2}^2.$$ 

According to Lemma 8 there exists some $p \in [H^1]^d$ such that

$$\text{div } p = \text{tr}(\sigma), \quad p = 0 \quad \text{on } \Gamma \setminus \Gamma_N, \quad \|p\|_1 \leq c \|\text{tr}(\sigma)\|_{L^2}.$$ 

For this $p$ we may write

$$\|\text{tr}(\sigma)\|_{L^2}^2 = \int_{\Omega} \text{tr}(\sigma) \text{ div}(p) \, dx$$

$$= \int_{\Omega} \sigma : (\text{div}(p) I) \, dx$$

$$= d \int_{\Omega} \sigma : (\nabla p - (\nabla p)^D) \, dx$$

$$= d \int_{\Omega} [\sigma : \nabla p \, dx - \sigma^D : \nabla p] \, dx$$

$$\leq d \int_{\Omega} \sigma : \nabla p \, dx + d\|\sigma^D\|_{L^2} \|p\|_{H^1}$$

$$\leq d \int_{\Omega} \sigma : \nabla p \, dx + d\|\sigma^D\|_{L^2} \|\text{tr}(\sigma)\|_{L^2}. $$
For the first term there holds
\[
\int_{\Omega} \sigma : \nabla p \, dx = \sum_T \int_T \text{div} \sigma \cdot p \, dx - \sum_F \int_F (\sigma n) \cdot p \, ds
\]
\[
= \sum_T \int_T \text{div} \sigma \cdot p \, dx - \sum_F \int_F [\sigma_{nr}] \cdot p_r \, ds
\]
\[
= \langle \text{div} \sigma, p \rangle
\]
\[
= \langle \text{div} \sigma, p - I_h p \rangle
\]
where we used that \( \langle \sigma, v_h \rangle = 0 \) for \( v_h \in V_h \) in the last line. Let \( I_V \) denote the Nédeléc interpolation operator, for which holds
\[
\int_E q(p - I_V p) \cdot \tau \, dx = 0 \quad \forall q \in P^1, E \in \mathcal{E}
\]
(14)
\[
\|p - I_V p\|_{L^2} \leq h\|p\|_{H^1}.
\]
(15)
On each face \( F \in \mathcal{F} \) \( p_r \) is in \( P^1 \). As the interpolation operator \( I_V \) is polynomial preserving, the face integrals in the estimate above vanish,
\[
\sum_F \int_F [\sigma_{nr}] (p_r - I_V p_r) \, ds = 0.
\]
Therefore we may estimate
\[
\int_{\Omega} \sigma : \nabla p \, dx = \sum_T \int_T \text{div} \sigma \cdot (p - I_V p) \, dx
\]
\[
\leq \sum_T \| \text{div} \sigma \|_{L^2(T)} \| p - I_V p \|_{L^2(T)}
\]
\[
\leq c \sum_T h \| \text{div} \sigma \|_{L^2(T)} \| p \|_1
\]
\[
\leq c \sum_T h \| \text{div} \sigma \|_{L^2(T)} \| \text{tr}(\sigma) \|_{L^2}.
\]
Collecting the results from above, we get
\[
\| \text{tr}(\sigma) \|_{L^2}^2 \leq c \sum_T h^2 \| \text{div} \sigma \|_{L^2(T)}^2 + d \| \sigma^D \|_{L^2}^2.
\]
For the bilinear form \( a^* \), this implies
\[
a^*(\sigma, \sigma) = \int_{\Omega} \frac{1}{2\mu} |\sigma^D|^2 + \frac{1}{\lambda + \mu} \text{tr}(\sigma)^2 \, dx + \sum_T h^2 \int_T |\text{div} \sigma|^2 \, dx
\]
\[
\geq \frac{1}{2\mu} \|\sigma^D\|^2_{L^2} + \sum_T h^2 \int_T |\text{div} \sigma|^2_{L^2(T)} \, dx
\]
\[
\geq c(\mu) \| \text{tr}(\sigma) \|_{L^2}^2.
\]
Remark 10. In the case of Dirichlet boundary conditions, it is not possible to find \( p \in H^1(\Omega)^d \) such that

\[
\begin{align*}
\text{div} p &= \text{tr}(\sigma), \\
p &= 0 \quad \text{on } \Gamma \\
\|p\|_{H^1} &\leq c \|\text{tr}(\sigma)\|_{L^2},
\end{align*}
\]

as \( \text{tr}(\sigma) \) does not necessarily satisfy the complementarity condition

\[
\int_{\Omega} \text{tr}(\sigma) \, dx = 0.
\]

In this case, we choose a different norm for the discrete space \( \Sigma_h \)

\[
\|\sigma\|_{\Sigma_h}^2 = \min_{\rho \in \mathbb{R}} \|\sigma - \rho I\|_{L^2}^2 + \frac{1}{\lambda} \|\rho I\|_{L^2}^2.
\]

We can find \( p \in H^1(\Omega)_0^d \) such that

\[
\begin{align*}
\text{div} p &= \text{tr}(\sigma) - \overline{\text{tr}(\sigma)}, \\
p &= 0 \quad \text{on } \Gamma \\
\|p\|_1 &\leq c \|\text{tr}(\sigma)\|_{L^2},
\end{align*}
\]

In this case we can bound

\[
\|\text{tr}(\sigma) - \overline{\text{tr}(\sigma)}\|_{L^2} \leq c \sum_T h_T \|\text{div} \sigma\|_{L^2(T)} + d \|\sigma^D\|_{L^2}
\]

Setting \( \rho = \frac{1}{d} \text{tr}(\sigma) \) we may estimate

\[
\begin{align*}
\|\sigma\|_{\Sigma_h}^2 &\leq \|\sigma - \rho I\|_{L^2}^2 + \frac{1}{\lambda} \|\rho I\|_{L^2}^2 \\
&\leq c \left( \|\sigma - \frac{1}{d} \text{tr}(\sigma) I\|_{L^2}^2 + \frac{1}{d^2} \|\text{tr}(\sigma) - \overline{\text{tr}(\sigma)}\|_{L^2}^2 \right) \\
&\leq c \left( \|\sigma^D\|_{L^2}^2 + \sum_T h_T^2 \|\text{div} \sigma\|_{L^2(T)}^2 \right) \\
&\leq c(\mu) a^*(\sigma, \sigma).
\end{align*}
\]

This proves stability for the Dirichlet case.

By the stability of the discrete saddle point problem and standard interpolation error estimates we obtain the convergence result

Theorem 11. The finite element solution \((u_h, \sigma_h)\) satisfies the error estimate

\[
\|u - u_h\|_{V_h} + \|\sigma - \sigma_h\|_{\Sigma_h} \leq ch^{m-1} \|u\|_{H^m}
\]
for \( u \in \mathcal{H}^m \) with \( 1 \leq m \leq k + 1 \). The constant \( c \) is bounded uniformly for \( \lambda \to \infty \).

**Remark 12** (Hybridization). Discretization of the mixed system leads to an indefinite system matrix. To avoid this, the continuity of \( \sigma_{nm} \) can be broken and enforced by inter-element Lagrange multipliers [16, 5, 14]. Then all degrees of freedom for the finite element space \( \Sigma_h \) are bound to only one element, and can be eliminated by static condensation. The remaining matrix is symmetric and positive definite. The Lagrange multipliers can be interpreted as normal displacements on the edges of the triangles or faces of the tetrahedra, respectively.

### 4. Finite element basis functions

To construct finite elements for the spaces \( \mathcal{V}_h \subset H(\text{curl}) \) and \( \Sigma_h \subset H(\text{div div}) \), we use the 2d-sequence

\[
\begin{align*}
\mathcal{H}^1 \cap [H^1(T)]^2 & \xrightarrow{\nabla} H(\text{curl}) \xrightarrow{\sigma_T} H(\text{div div}) \xrightarrow{\text{div}} H^{-1}(\text{div}) \xrightarrow{\text{div}} H^{-1}
\end{align*}
\]

and in 3d

\[
\begin{align*}
\mathcal{H}^1 \cap [H^1(T)]^2 & \xrightarrow{\nabla} H(\text{curl}) \xrightarrow{\varepsilon_T} H(\text{curl curl}) \xrightarrow{\text{curl}} H(\text{div curl}) \\
& \xrightarrow{\text{sym curl} Q^{-1}} H(\text{div div}) \xrightarrow{\text{div}} H^{-1}(\text{div}) \xrightarrow{\text{div}} H^{-1}
\end{align*}
\]

with the operator \( Q \), its inverse \( Q^{-1} \) and the stress operator \( \sigma \) given by

\[
QA = A^T - \text{tr}(A)I, \quad Q^{-1}A = A^T - \frac{1}{d-1} \text{tr}(A)I, \quad \sigma = Q^{-1}\varepsilon
\]

The additional spaces used in the sequence are

\[
\begin{align*}
\mathcal{H}(\text{curl curl}) & := \{ \varepsilon \in [L^2(\Omega)]^{3\times3}^\text{SYM} : \text{sym curl } Q^{-1}\varepsilon \in [H^{-1}]^\text{SYM} \} \\
\mathcal{H}(\text{div curl}) & := \{ \delta \in [L^2(\Omega)]^{3\times3} : \text{div sym curl } Q^{-1}\delta \in [H^{-1}]^3 \}
\end{align*}
\]

The image of two successive operators is exactly the kernel of the next, in 2d there holds

\[
\begin{align*}
\text{range}(\sigma(\nabla \cdot)) & = \text{Ker}(\text{div}) \\
\text{range}(\text{div } \sigma(\cdot)) & = \text{Ker}(\text{div})
\end{align*}
\]

In 3d, we use the relation

\[
\text{range}(\text{sym curl } Q^{-1}\text{curl}) = \text{Ker}(\text{div})
\]
For the construction of shape functions on the reference elements, we use Legendre polynomials \( (\ell_i)_{0 \leq i \leq k} \), which are orthogonal polynomials in \( L_2(-1,1) \) and span \( P^k[-1,1] \). The integrated Legendre polynomials \( (L_i)_{2 \leq i \leq k} \) are defined by

\[
L_i(x) = \int_{-1}^{x} \ell_{i-1}(\xi) \, d\xi.
\]

They are orthogonal with respect to the \( H^1 \) semi norm and vanish on the interval bounds \( \{-1, 1\} \). We will also need scaled Legendre polynomials \( (\ell_{i}^{S})_{1 \leq i \leq k} \) and scaled integrated Legendre polynomials \( (L_{i}^{S})_{2 \leq i \leq k} \), which are defined for \( 0 < t \leq 1 \) and \( -t \leq x \leq t \). They are given by

\[
\ell_{i}^{S}(x, t) = t^{i} \ell_{i}(\frac{x}{t})
\]

\[
L_{i}^{S}(x, t) = t^{i} L_{i}(\frac{x}{t}).
\]

They satisfy \( L_{i}^{S}(-t, t) = L_{i}^{S}(t, t) = 0 \).

4.1. Triangular elements. Let the reference triangle \( T \) be defined by

\[
T := \{(x, y) \mid 0 \leq x, y \leq 1, \ x + y < 1\}
\]

with vertices \( V_1 = (1, 0), V_2 = (0, 1), V_2 = (0, 0) \). Let \( \lambda_i, i = 1, 2, 3 \) denote the barycentric coordinates

\[
\lambda_1 = x, \lambda_2 = y, \lambda_3 = 1 - x - y.
\]

For \( i, j \geq 0 \), let

\[
u_{i} = L_{i+2}^{S}(\lambda_2 - \lambda_1, \lambda_2 + \lambda_1)
\]

\[
\nu_{j} = \lambda_3 \ell_{j}(2\lambda_3 - 1).
\]

Then \( \nu_{i} \) is a polynomial of degree \( i + 2 \), which vanishes on edges \( E_1, E_2 \). The polynomial \( \nu_{j} \) is of degree \( j + 1 \) and vanishes on edge \( E_3 \).

The following \( H^1 \) conforming shape functions form a basis for \( P^k(T) \) [24, 28]

- Vertex-based functions: \( m = 1, 2, 3 \)

\[
\phi_{m}^{V} = \lambda_i
\]

- Edge-based functions: \( m = 1, 2, 3, \ E_{m} = [\alpha, \beta], \ 0 \leq i \leq k - 2 \)

\[
\phi_{E_{m}}^{E} = L_{i+2}^{S}(\lambda_{\alpha} - \lambda_{\beta}, \lambda_{\alpha} + \lambda_{\beta})
\]

- Cell-based functions: \( i, j \geq 0, i + j \leq k - 3 \)

\[
\phi_{i,j}^{T} = \nu_{i} \nu_{j} = L_{i-2}^{S}(\lambda_2 - \lambda_1, \lambda_2 + \lambda_1) \lambda_3 \ell_{j}(2\lambda_3 - 1)
\]
We define the local space of order $k$

$$W_k(T) = \text{span} (\phi^V_m) \oplus \text{span} (\phi^E_m) \oplus \text{span} (\phi^T_{i,j}) = P^k(T).$$

Using this, $H(\text{curl})$ conforming shape functions can be defined as below [24, 28]

- **Edge-based functions**: $m = 1, 2, 3$
  - Low-order functions: $m = 1, 2, 3$, $E_m = [\alpha, \beta]$
    $$\varphi_m^N = \nabla\lambda_\alpha\lambda_\beta - \lambda_\alpha\nabla\lambda_\beta$$
  - High-order functions: $0 \leq i \leq k - 1$
    $$\varphi^E_m = \nabla\phi^E_m$$

- **Cell-based functions**:
  - Type 1, gradient fields: $0 \leq i + j \leq k - 2$
    $$\varphi^{T,1}_{i,j} = \nabla(u_i v_j) = \nabla u_i v_j + u_i \nabla v_j$$
  - Type 2, $0 \leq i + j \leq k - 2$
    $$\varphi^{T,2}_{i,j} = \nabla u_i v_j - u_i \nabla v_j$$
  - Type 3, $0 \leq j \leq k - 2$
    $$\varphi^{T,3}_j = (\nabla\lambda_1\lambda_2 - \lambda_1\nabla\lambda_2) v_j$$

The local $H(\text{curl})$ conforming FE-space is

$$V_k(T) = \text{span} (\varphi^N_m) \oplus \text{span} (\varphi^E_m) \oplus \text{span} (\varphi^T_{i,j}) = P^k(T).$$

There holds

$$\nabla W_{k+1}(T) \subset V_k(T).$$

As a last step, we define $H(\text{div div})$ conforming shape functions on the triangle $T$

- **Edge-based functions**, divergence free:
  $m = 1, 2, 3$, $E_m = [\alpha, \beta]$, $0 \leq i \leq k$
  $$\psi^E_m = \sigma(\nabla\phi^E_m)$$

- **Cell-based functions**:
  - Type 1, divergence free:
    $i, j \geq 0$, $0 \leq i + j \leq k - 1$
    $$\psi^{T,1}_{i,j} = \sigma(\nabla (u_i v_j)) = \nabla^2(u_i v_j) - \Delta(u_i v_j) I$$
    $$= (\nabla^2 u_i v_j + \nabla u_i \nabla v_j^T + \nabla v_j \nabla u_i^T + u_i \nabla^2 v_j)$$
    $$- (\Delta u_i v_j + 2 \nabla u_i^T \nabla v_j + u_i \Delta v_j) I$$
- Type 2, $i, j \geq 0$, $0 \leq i + j \leq k - 1$
\[ \psi^{T,2}_{i,j} = (\nabla^2 u_i v_j - \nabla u_i \nabla v_j^T - \nabla v_j \nabla u_i^T + u_i \nabla^2 v_j) - (\Delta u_i v_j - 2\nabla u_i^T \nabla v_j + u_i \Delta v_j)I \]
- Type 3, $i \geq 0$, $j \geq 1$, $1 \leq i + j \leq k - 1$
\[ \psi^{T,3}_{i,j} = (\nabla^2 u_i v_j - u_i \nabla^2 v_j) - (\Delta u_i v_j - u_i \Delta v_j)I \]
- Type 4, $0 \leq j \leq k - 1$
\[ \psi^{T,4}_j = \sigma((\nabla \lambda_1 \lambda_2 - \lambda_1 \nabla \lambda_2) v_j) \]

From these basis functions, we derive the local $H(\text{div} \text{ div})$ conforming FE-space
\[ \Sigma_k(T) = \text{span}(\psi^{F_m}_i) \oplus \text{span}(\psi^{T,i}_{i,j}) = P^k(T). \]

The way the basis was constructed implies
\[ \sigma(V_{k+1}(T)) \subset \Sigma_k(T). \]

4.2. Tetrahedral elements. Let the reference tetrahedron $T$ be defined by
\[ T := \{(x, y, z) \mid 0 \leq x, y, z \leq 1, \ x + y + z < 1\} \]
with vertices $V_1 = (1, 0, 0), V_2 = (0, 1, 0), V_3 = (0, 0, 1), V_4 = (0, 0, 0)$. Let $\lambda_i, i = 1 \ldots 4$ denote the barycentric coordinates
\[ \lambda_1 = x, \lambda_2 = y, \lambda_3 = z, \lambda_4 = 1 - x - y - z. \]

For $i, j \geq 0$, let
\[ u_i = L^S_{i+2}(\lambda_2 - \lambda_1, \lambda_2 + \lambda_1) \]
\[ v_j = \lambda_3 \ell^S_j(\lambda_3 - \lambda_2 - \lambda_1, \lambda_3 + \lambda_2 + \lambda_1) \]
\[ w_k = \lambda_4 \ell(2\lambda_4 - 1) \]

The polynomials $u_i$ vanish on faces $F_1, F_2$, whereas $v_j = 0$ on $F_3$ and $w_k = 0$ on $F_4$.

The following functions define a local basis for $H^1(T)$
- Vertex based functions, $m = 1 \ldots 4$
\[ \phi^m_m = \lambda_m \]
- Edge based functions, $m = 1 \ldots 6$, $E_m = [\alpha, \beta]$, $0 \leq i \leq k - 2$
\[ \phi^{E_m}_i = L^S_{i+2}(\lambda_\alpha - \lambda_\beta, \lambda_\alpha + \lambda_\beta) \]
- Face based functions, $m = 1 \ldots 4$, $F_m = [\alpha, \beta, \gamma]$, $0 \leq i + j \leq k - 3$
\[ u_i^{\alpha,\beta} = L^S_{i+2}(\lambda_\alpha - \lambda_\beta, \lambda_\alpha + \lambda_\beta), \ v_j^{\gamma} = \lambda_\gamma \ell^S(\lambda_\gamma - \lambda_\alpha - \lambda_\beta, \lambda_\gamma + \lambda_\alpha + \lambda_\beta) \]
\[ \phi^{F_m}_{i,j} = L^S_{i+2}(\lambda_\alpha - \lambda_\beta, \lambda_\alpha + \lambda_\beta) \lambda_\gamma \ell^S_j(2\lambda_\gamma - \lambda, \lambda) \]
• Cell based functions, $0 \leq i + j + l \leq k - 4$

$$\varphi_{i,j,l}^T = u_i v_j w_l$$

The local space of order $k$ is given by

$$W_k(T) = \text{span}(\varphi_{m}^N) \oplus \text{span}(\varphi_{i}^E) \oplus \text{span}(\varphi_{i,j}^F) \oplus \text{span}(\varphi_{i,j,l}^T) = P^k(T).$$

Again we use these functions to construct $H(\text{curl})$ conforming shape functions

• Edge-based functions, $m = 1 \ldots 6$, $E_m = [\alpha, \beta]$

  – Low-order functions:

  $$\varphi_{m}^{N0} = \nabla \lambda_\alpha \lambda_\beta - \lambda_\alpha \nabla \lambda_\beta$$

  – High-order functions: $0 \leq i \leq k - 1$

  $$\varphi_{m}^{E} = \nabla \varphi_{m}^{E}$$

• Face-based functions: $m = 1 \ldots 4$, $F_m = [\alpha, \beta, \gamma]$,

  $u_i^{\alpha,\beta} = L_{i+2}^S(\lambda_\alpha - \lambda_\beta, \lambda_\alpha + \lambda_\beta)$,
  $v_j^{\gamma} = \lambda_\gamma \xi^S(\lambda_\gamma - \lambda_\alpha - \lambda_\beta, \lambda_\gamma + \lambda_\alpha + \lambda_\beta)$

  – Type 1, $0 \leq i + j \leq k - 2$

  $$\varphi_{i,j}^{F1,m} = \nabla (\varphi_{i,j}^{m}) = \nabla (u_i^{\alpha,\beta} v_j^{\gamma})$$

  – Type 2, $0 \leq i + j \leq k - 2$

  $$\varphi_{i,j}^{F2,m} = \nabla u_i^{\alpha,\beta} v_j^{\gamma} - u_i^{\alpha,\beta} \nabla v_j^{\gamma}$$

  – Type 3, $0 \leq j \leq k - 2$

  $$\varphi_{j}^{F3,m} = (\nabla \lambda_\alpha \lambda_\beta - \lambda_\alpha \nabla \lambda_\beta) v_j^{\gamma}$$

• Cell-based functions:

  – Type 1, $0 \leq i + j + l \leq k - 3$

  $$\varphi_{i,j,l}^{T1} = \nabla (\varphi_{i,j,l}^{T}) = \nabla (u_i v_j w_l)$$

  – Type 2, $0 \leq i + j + l \leq k - 3$

  $$\varphi_{i,j,l}^{T2a} = \nabla u_i v_j w_l - u_i \nabla v_j w_l + u_i v_j \nabla w_l$$

  $$\varphi_{i,j,l}^{T2b} = \nabla u_i v_j w_l + u_i \nabla v_j w_l - u_i v_j \nabla w_l$$

  – Type 3, $0 \leq j + l \leq k - 3$

  $$\varphi_{j,l}^{T3} = (\nabla \lambda_1 \lambda_2 - \lambda_1 \nabla \lambda_2) v_j w_l$$

We define the local $H(\text{curl})$ conforming FE-space

$$V_k(T) = \text{span}(\varphi_{m}^{N0}) \oplus \text{span}(\varphi_{i}^{E}) \oplus \text{span}(\varphi_{i,j}^{F}) \oplus \text{span}(\varphi_{i,j,l}^{T}) = P^k(T).$$

As in 2D, there holds

$$\nabla W_{k+1}(T) \subset V_k(T).$$

$H(\text{div div})$ conforming shape functions on the tetrahedron $T$ can be constructed as follows
• Face-based functions, divergence free
  \[ m = 1 \ldots 4, F_m = [\alpha, \beta, \gamma], \]
  \[ u_i^{\alpha,\beta} = L_{i+2}^S(\lambda_\alpha - \lambda_\beta, \lambda_\alpha + \lambda_\beta), \quad v_j^\gamma = \lambda_\gamma \ell^S(\lambda_\gamma - \lambda_\alpha - \lambda_\beta, \lambda_\gamma + \lambda_\alpha + \lambda_\beta) \]
  \[ 1 \leq i + j \leq k + 2 \]
  \[ \psi_{i,j}^{Fm,1} = \text{sym} \left( \text{curl}(\nabla u_i^{\alpha,\beta} \nabla v_j^\gamma)^T \right) \]

• Cell-based functions:
  - Type 1, divergence free,
    \[ 0 \leq i + j + l \leq k \]
    \[ \psi_{i,j,l}^{T1a} = \text{sym} \left( \text{curl}(u_i \nabla v_j \nabla w_l)^T \right) \]
    \[ \psi_{i,j,l}^{T1b} = \text{sym} \left( \text{curl}(v_j \nabla u_i \nabla w_l)^T \right) \]
    \[ \psi_{i,j,l}^{T1c} = \text{sym} \left( \text{curl}(w_l \nabla v_j \nabla u_i)^T \right) \]
  - Type 2, \[ 0 \leq i + j + l \leq k \]
    \[ \psi_{i,j,l}^{T2a} = \text{sym} \left( \nabla v_j \times (\nabla^2 u_i) \times \nabla w_l - u_i \text{curl}(\nabla v_j \nabla w_l)^T \right), \quad j, l \geq 1 \]
    \[ \psi_{i,j,l}^{T2b} = \text{sym} \left( \nabla u_i \times (\nabla^2 v_j) \times \nabla w_l - v_j \text{curl}(\nabla u_i \nabla w_l)^T \right), \quad l \geq 1 \]
    \[ \psi_{i,j,l}^{T2c} = \text{sym} \left( \nabla v_j \times (\nabla^2 w_l) \times \nabla u_i - w_l \text{curl}(\nabla v_j \nabla u_i)^T \right), \quad j \geq 1 \]
  - Type 3, \[ 0 \leq i + j \leq k \]
    \[ \psi_{i,j}^{T3a} = \text{sym} \left( \nabla v_i \times (\nabla \phi_{N_0}^{E_{i+2}}) \times \nabla w_j \right) \]
    \[ \psi_{i,j}^{T3b} = \text{sym} \left( \phi_{N_0}^{E_{i+2}} \times (\nabla^2 v_i) \times \nabla w_j \right), \quad i \geq 1 \]

The local \( H(\text{curl}) \) conforming FE-space is defined by

\[ \Sigma_k(T) = \text{span}(\psi_{i,j}^{Fm}) \oplus \text{span}(\psi_{i,j,l}^T) = P^k(T). \]

Similar to the 2D case, we have

\[ \sigma(V_{k+1}(T)) \subset \Sigma_k(T). \]

4.3. **Global Finite Element Spaces.** The global finite element spaces \( W_h \subset H^1, V_h \subset H(\text{curl}) \) and \( \Sigma_h \subset H(\text{div div}) \) can be derived from the local spaces on the reference elements by element-wise transformation. Therefore, let \( \Phi_T : \hat{T} \rightarrow T \) be the mapping from the reference element \( \hat{T} \) to the physical element \( T \). By \( F_T \) we denote the Jacobian of this transformation, \( F_T(\hat{x}) = \left[ \frac{\partial \Phi_{T,i}}{\partial \hat{x}_j}(\hat{x}) \right]_{ij} \). The Jacobian determinant is given by \( J_T(\hat{x}) = \det(F_T(\hat{x})) \). Normal and tangential vectors along the element boundary transform as

\[ \tau = F_T^T \hat{\tau}, \quad n = J_T F_T^{-T} \hat{n}. \]
An $H^1$ conforming transformation from $\hat{T}$ to $T$ is given by

$$H^1(\hat{T}) \rightarrow H^1(T), \quad \hat{w} \mapsto w := \hat{w} \circ \Phi_T^{-1}.$$ 

For $H(\text{curl})$, we use the covariant transformation is [21]

$$H(\text{curl}, \hat{T}) \rightarrow H(\text{curl}, T), \quad \hat{v} \mapsto v := F_T^{-T} \hat{v} \circ \Phi_T^{-1}.$$ 

The covariant transformation is conforming for $H(\text{curl})$, as the tangential component of a vector field along the boundary of the element is preserved.

The following transformation is conforming for $H(\text{div div})$

$$H(\text{div div}, \hat{T}) \rightarrow H(\text{div div}, T), \quad \hat{\sigma} \mapsto \sigma := \frac{1}{J_T^2} (F_T \hat{\sigma} F_T^T) \circ \Phi_T^{-1}.$$ 

For this transformation, the normal-normal component of the stress field is preserved,

$$n^T \sigma n = \frac{1}{J_T^2} n^T F_T^{-1} F_T \sigma F_T^T F_T^{-T} n = \hat{n}^T \hat{\sigma} \hat{n}.$$ 

Using these transformations, and the local finite element spaces $W_h(\hat{T})$, $V_h(\hat{T})$ and $\Sigma_h(\hat{T})$ defined above, we get finite element spaces $W_h, V_h$ and $\Sigma_h$ for $H^1, H(\text{curl})$ and $H(\text{div div})$, respectively.

5. Non-conforming elements

Until now we have considered finite dimensional spaces $\Sigma_h \subset \Sigma = H(\text{div div})$ and $V_h \subset V = H(\text{curl})$. A piecewise smooth function $v$ is in $H(\text{curl})$, if and only if the tangential component $v_\tau$ is continuous across element interfaces. In 2D, there are $k + 1$ degrees of freedom on each edge, which are associated to the tangential displacement along this edge. In this section, we will not assume that the finite element space $V_h$ is a subset of $H(\text{curl})$, but that the highest order edge basis function may jump across element edges. Thereby we obtain a simplified element. It consists of more internal degrees of freedom, which can be eliminated locally.

Let $(\sigma, u) \in \Sigma \times V$ denote the solution to the continuous problem, and $(\sigma_h, u_h) \in \Sigma_h \times V_h$ the finite-element solution, where $V_h$ is the non-conforming space described above. We will see that the error

$$\| (\sigma - \sigma_h, u - u_h) \|_{\Sigma_h \times V_h}$$
can be bounded by the same order of convergence as in the conforming case. The triangle inequality ensures
\[ \|(\sigma - \sigma_h, u - u_h)\|_{\Sigma_h \times V_h} \leq \|(\sigma - I_\Sigma \sigma, u - I_V u)\|_{\Sigma_h \times V_h} + \|(I_\Sigma \sigma - \sigma_h, I_V u - u_h)\|_{\Sigma_h \times V_h} \]
for suitable interpolation operators $I_V, I_\Sigma$. For the first term we know, provided the solution is sufficiently smooth
\[ \|(\sigma - I_\Sigma \sigma, u - I_V u)\|_{\Sigma_h \times V_h} \leq c h^k \|(\sigma, u)\|_{H^k \times H^{k+1}}. \]
For the second term, we use Galerkin orthogonality with respect to $\sigma$, as $\Sigma_h \subset \Sigma$.
\[ \|(I_\Sigma \sigma - \sigma_h, I_V u - u_h)\|_{\Sigma_h \times V_h} \leq \bar{\beta} \sup_{\tau_h, v_h} \frac{B(I_\Sigma \sigma - \sigma_h, I_V u - u_h; \tau_h, v_h)}{\|(\tau_h, v_h)\|_{\Sigma_h \times V_h}} \]
\[ \leq \bar{\beta} \sup_{\tau_h, v_h} \frac{B(I_\Sigma \sigma - \sigma, I_V u - u; \tau_h, v_h)}{\|(\tau_h, v_h)\|_{\Sigma_h \times V_h}} + \bar{\beta} \sup_{\tau_h, v_h} \frac{B(\sigma - \sigma_h, u - u_h; \tau_h, v_h)}{\|(\tau_h, v_h)\|_{\Sigma_h \times V_h}} \]
Again, the first term can be bounded by the approximation error and is of order $h^k$. For the second term, we use Galerkin orthogonality with respect to $\sigma$, as $\Sigma_h \subset \Sigma$.
\[ \sup_{\tau_h, v_h} \frac{B(\sigma - \sigma_h, u - u_h; \tau_h, v_h)}{\|(\tau_h, v_h)\|_{\Sigma_h \times V_h}} = \sup_{v_h} \frac{B(\sigma - \sigma_h, u - u_h; 0, v_h)}{\|v_h\|_{V_h}} = \sup_{v_h} \frac{b(\sigma - \sigma_h, v_h)}{\|v_h\|_{V_h}} \]
(16)
The edge basis functions associated to the tangential displacement are continuous for order 0 to $k - 1$, only the highest order function jumps.
Therefore, \( v_h \in V_h \) can be divided into a conforming part \( v^c \), which consists of the lower-order edge basis functions and the internal basis functions, and into the non-conforming part. From the construction of the basis for \( V_h \) we know that the restriction of this non-conforming part to a triangle \( T \) is the gradient of an \( H^1 \) edge bubble function of order \( k + 1 \). We may write

\[
\begin{align*}
v_h &= v^c + \nabla_h \phi \\
v^c &\in H(\text{curl}) \\\\n\nabla_h \phi |_T &= \sum_{i=1}^{3} r_{T,i} \nabla \phi_{E_i} |_T \quad \forall T \in T.
\end{align*}
\]

We use this and the fact that \( \sigma_h \) is a solution to the discrete problem to obtain

\[
\begin{align*}
b(\sigma - \sigma_h, v_h) &= b(\sigma, v_h) - (f, v_h) \\
&= b(\sigma, v^c) - (f, v^c) + b(\sigma, \nabla_h \phi) - (f, \nabla_h \phi) \\
&= \sum_T \int_T (\text{div} \sigma - f) \cdot \nabla \phi \, dx - \int_{\partial T} \sigma_{n_T} \cdot \frac{\partial \phi_h}{\partial \tau} \, ds \\
&= \sum_T \int_{\partial T} \sigma_{n_T} \cdot \frac{\partial \phi_h}{\partial \tau} \, ds.
\end{align*}
\]

Therefore, we can estimate the right hand side of equation (16) by

\[
\begin{align*}
\sup_{v_h} \frac{b(\sigma - \sigma_h, v_h)}{\|v_h\|_{V_h}} &\leq \sup_{\phi_h} \frac{b(\sigma - \sigma_h, \nabla_h \phi_h)}{\|\nabla_h \phi_h\|_{V_h}} \\
&= \sup_{\phi_h} \frac{\sum_T \int_{\partial T} \sigma_{n_T} \cdot \frac{\partial \phi_h}{\partial \tau} \, ds}{\sum_T \|\nabla^2 \phi_h\|_{L^2(T)}^2 + h^{-1} \|\frac{\partial \phi_h}{\partial n}\|_{L^2(\partial T)}^2}^{1/2}
\end{align*}
\]

For an edge bubble function \( \phi_h \in P^{k+1}(T) \) there holds

\[
\|\nabla^2 \phi_h\|_{L^2(T)}^2 + h^{-1} \|\frac{\partial \phi_h}{\partial n}\|_{L^2(\partial T)}^2 \simeq ch^{-2} \|\phi_h\|_{L^2(T)}.
\]

Moreover, \( \phi_h \) restricted to an edge is an integrated Legendre polynomial of degree \( k + 1 \). This implies that \( \partial \phi_h / \partial \tau \) is orthogonal to polynomials
of degree $k - 1$, and

$$
\sup_{v_h} \frac{b(\sigma - \sigma_h, v_h)}{\|v_h\|_{V_h}} \leq c \sup_{\phi_h} \sum_{T:F \subset \partial T} \inf_{s \in P^{k-1}} \frac{\int_F (\sigma_{nt} - s) \frac{\partial \phi_h}{\partial t} \, ds}{h^{-2} \|\phi_h\|_{L^2}}
$$

$$
\leq c \sup_{\phi_h} \sum_{T:F \subset \partial T} \inf_{s \in P^{k-1}} \frac{\|\sigma_{nt} - s\|_{L^2(F)} \|\frac{\partial \phi_h}{\partial t}\|_{L^2(F)}}{h^{-2} \|\phi_h\|_{L^2}}
$$

$$
\leq c \sup_{\phi_h} \sum_{T:F \subset \partial T} \inf_{s \in P^{k-1}} \frac{\|\sigma_{nt} - s\|_{L^2(F)} h^{-3/2} \|\phi_h\|_{L^2(T)}}{h^{-2} \|\phi_h\|_{L^2}}
$$

$$
\leq c \sum_{T,F \subset \partial T} h^{k+1/2} \|\sigma_{nt}\|_{H^k(F)}
$$

$$
\leq c h^k \|\sigma\|_{H^k(\Omega)}.
$$

This implies an optimal order of convergence of $\sigma$ in the $L_2$ norm for the 2D non-conforming elements.

### 6. Anisotropic Estimates

Many technical constructions involve thin structures such as beams, plates or shells. Traditionally lower dimensional models are derived and treated by proper finite element methods. Here, the small thickness parameter becomes explicit, and special care must be taken to avoid the so-called shear locking.

The concept of hierarchical finite element modeling treats the structure directly by anisotropic three-dimensional finite elements, and the accuracy of the model corresponds to the polynomial order in thickness direction [11, 3, 17, 20], a posteriori estimates are in [25, 2].

Anisotropic error estimates for scalar equations are developed in [4]. Due to the combination of derivatives of the strain operator, these techniques cannot be applied to conforming finite elements for elasticity problems. The goal of this section is to derive robust anisotropic estimates for the elasticity problem discretized with the proposed new elements.

Now, let $\omega \subset \mathbb{R}^2$ be a domain identified with the cross-section of a flat three-dimensional domain, let $I = (0, t)$, and set $\Omega = \omega \times I$. Furthermore, let $T^{xy}$ and $T^z$ be triangulations of $\omega$ and $I$, respectively. Then $T^{xy} \otimes T^z$ gives the tensor-product mesh of the three-dimensional domain. We may use just one element in thickness direction, or we may use also a refined mesh to resolve several layers of a laminated plate.
We define Lagrangian and Nédelec finite element spaces on the cross section and on the interval as

\[
L_{xy}^k := \{ v \in H^1(\omega) : v_T \in P^k \},
\]
\[
N_{xy}^k := \{ v \in H(\text{curl},\omega) : v_T \in [P_k]^2 \},
\]
\[
\mathcal{P}_{xy}^k := \{ v \in L^2(\omega) : v_T \in P^k \},
\]
\[
\mathcal{L}_z^k := \{ v \in H^1(I) : v_T \in P^k \},
\]
\[
\mathcal{P}_z^k := N_z^k := \{ v \in L^2(I) : v_T \in P^k \}.
\]

The spaces satisfy \( \nabla L_{xy}^{k+1} \subset N_{xy}^k \) and \( \nabla L_z^{k+1} \subset N_z^k \). Note that \( V_h := N_{xy}^k \otimes L_z^{k+1} \times L_{xy}^{k+1} \otimes N_z^k \) gives the Nédelec space on the tensor product domain. Let \( \Sigma_{xy}^k \) be the finite element space of symmetric \( 2 \times 2 \) tensors with continuous normal-normal components, and set

\[
\Sigma_h := \left\{ \sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix} : \left( \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix} \right) \in \Sigma_{xy}^k \otimes N_z^k, \\
\left( \begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \end{pmatrix} \right) \in P_{xy}^k \otimes N_z^k, \sigma_{zz} \in P_{xy}^k \otimes L_z^{k+2} \right\}.
\]

By [15] there exist quasi-interpolation operators \( I_{xy}^k : L_2(\omega) \to L_{xy}^k \) such that

\[
\|u - I_{xy}^k u\|_{l,T} \lesssim h^{m-l}\|u\|_{m,\tilde{T}} \quad 0 \leq l \leq 1, 0 \leq m \leq k+1, l \leq m,
\]

where \( \tilde{T} \) is the union of the triangle \( T \) and its neighbours. In [23] quasi-interpolation operators \( Q_{xy}^k : L_2(\omega) \to N_{xy}^k \) for the Nédelec spaces have been introduced. They commute in the sense of

\[
\nabla I_{xy}^k = Q_{xy}^k \nabla,
\]

and satisfy a corresponding interpolation error estimate

\[
\|u - Q_{xy}^k u\|_{l,T} \lesssim h^{m-l}\|u\|_{m,\tilde{T}} \quad 0 \leq l \leq 1, 0 \leq m \leq k, l \leq m.
\]

Note that the estimates for \( \|\cdot\|_{1,T} \) are not contained in [23], but are also easily proven. We define according commuting interpolation operators in thickness direction. On the tensor product domain we define the interpolation operator

\[
Q_k := Q_{xy}^k \otimes I_{z}^{k+1} \times I_{xy}^{k+1} \otimes Q_z^k.
\]
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The in-plane deformation \( u_{xy} := (u_x, u_y) \) is interpolated by the Nédélec operator in plane, and continuously in the thickness-direction, the transversal displacement \( u_z \) is interpolated vice versa.

**Lemma 13.** The product space interpolation operator \( Q_k \) satisfies the anisotropic error estimate

\[
(17) \quad \| \varepsilon(u - Q_k u) \|_{L_2(T)} \leq h_{xy}^m \| \nabla_{xy}^m \varepsilon(u) \|_{L_2(T)} + h_z^m \| \nabla_z^m \varepsilon(u) \|_{L_2(\Gamma)}
\]

**Proof.** We bound the different components of the strain tensor as follows:

\[
\| \varepsilon_{xy}(u_{xy} - I_{k+1}z Q_k^z u_{xy}) \|_T \leq \| \varepsilon_{xy}(u_{xy} - Q_k^z u_{xy}) \|_T + \| \varepsilon_{xy}(Q_k^z u_{xy} - I_{k+1}z u_{xy}) \|_T
\]

\[
\leq h_{xy}^m \| \nabla_{xy}^m \varepsilon_{xy}(u_{xy}) \|_{\tilde{T}} + \| \varepsilon_{xy}(u_{xy} - I_{k+1}z u_{xy}) \|_{\tilde{T}}
\]

The second term \( \varepsilon_{xy,z} \) involves the coupling of in-plane and transversal deformations and essentially relies on the commuting diagram:

\[
2 \| \varepsilon_{xy,z}(u - Q_k u) \|_{0,T} = \| \nabla_z(u_{xy} - Q_k^z I_{k+1}z u_{xy}) + \nabla_{xy}(u_{xy} - Q_k^z I_{k+1}z u_{xy}) \|
\]

\[
= \| (I - Q_k^z I_{k+1}z) (\nabla_z u_{xy} + \nabla_{xy} u_z) \|
\]

\[
\leq h_{xy}^m \| \nabla_{xy}^m \varepsilon_{xy,z}(u) \|_{\tilde{T}} + h_z^m \| \nabla_z^m \varepsilon_{xy,z}(u) \|_{\tilde{T}}
\]

The last term, \( \varepsilon_z(u_z - Q_k^z I_{k+1}z u_z) \) is bounded similar to the first one.

\[\square\]

Anisotropic estimates for the stress tensor are also simply obtained. A careful investigation of the estimates in Section 3 provides that all stability estimates are independent of the aspect ratio of the elements. For this, the weight of the jump-terms must be chosen as the element size norm to the facet. Since the norm equivalence \( V_h \simeq \tilde{V}_h \) of Lemma 5 does not hold anymore, \( \sigma_{nn} \) must be chosen at the same order as \( [u_n] \).

With the same arguments as in the isotropic case, we obtain the

**Theorem 14** (anisotropic error estimates). Let \( u_h \) and \( \sigma_h \) be the finite element solutions on an anisotropic finite element mesh. Then there holds the anisotropic error estimate

\[
(18) \quad \| u - u_h \|_{V_h} + \| \sigma - \sigma_h \|_{L_2} \leq c h_{xy}^m \| \nabla_{xy}^m \varepsilon(u) \|_{\Omega} + c h_z^m \| \nabla_z^m \varepsilon(u) \|_{\Omega}
\]

The constant \( c \) is independent of the aspect ratio of the elements.
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1e-05
1e-04
0.001
0.01
0.1
1
10  100  1000  10000  100000  1e+06  1e+07
error stress
dofs
order 1
order 1, nc
order 2
order 2, nc

Figure 6. Unit square, error $\|\sigma - \sigma_h\|_{L^2(\Omega)}$, $\nu = 0.3$

7. Numerical Results

The first example we consider is the unit square. We suppose the left side is fixed, all other boundaries are free. There is a constant body force acting in vertical direction. We assume that the material is homogenous, isotropic, linear elastic. We calculate stresses $\sigma_h$ and displacements $u_h$ using the method and elements described above. We start from a coarse mesh, which is refined adaptively. As an error estimator for the refinement we use a Zienkiewicz-Zhou type estimator.

For Figure 7, the material parameters are given by Young’s modulus $E = 1$ and the Poisson ratio $\nu = 0.3$. We plot the decrease of the $L^2$-error of the stresses, $\|\sigma - \sigma_h\|_{L^2(\Omega)}$, as the number of unknowns grows. We use both first and second order elements, and observe an optimal order of convergence. The same behavior can be seen for the non-conforming elements described in Section 5.

The elements do not suffer from volume locking, as the material gets nearly incompressible. In Figure 7, we plot the $L^2$-error of the stresses for a Poisson ratio $\nu = 0.4999$. Again, we see the optimal order convergence.

As a second example we consider a U-domain, where the left top is fixed, and a surface load is acting on the right top. The domain is discretized by 26 elements. We use curved elements of order 5 to approximate the circular part of the U-domain. Also the finite element spaces $\Sigma_h, V_h$ are 5th order. In Figure 7 we plot the $xx$-component of the stress tensor.

The last example is a cylindrical shell, where the radius of the cylinder is $R = 0.5$. The thickness of the shell is given as $t = 0.005$. In Figure 7, we used $P^1$ elements for the stresses $\sigma$ and $P^2$ elements for
Figure 7. Unit square, nearly incompressible ($\nu = 0.4999$)

Figure 8. U-domain, $\sigma_{xx}$

Figure 9. Cylindrical shell, $\sigma_{xy}$, $\sigma \in P^1$, $u \in P^2$

$u$. We plot the shear stress $\sigma_{xy}$. In Figure 7, we increase the order by one, i.e. $P^2$ for $\sigma_h$ and $P^3$ for $u_h$. 
Figure 10. Cylindrical shell, $\sigma_{xy}$, $\sigma \in P^2$, $u \in P^3$

References


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