Rational Solutions of the Painlevé Equations, Associated Special Polynomials and Applications to Soliton Equations. I

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Painlevé Equations

\[
\begin{align*}
\frac{d^2 w}{dz^2} &= 6w^2 + z & \text{P}_I \\
\frac{d^2 w}{dz^2} &= 2w^3 + zw + \alpha & \text{P}_{II} \\
\frac{d^2 w}{dz^2} &= \frac{1}{w} \left( \frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w} & \text{P}_{III} \\
\frac{d^2 w}{dz^2} &= \frac{1}{2w} \left( \frac{dw}{dz} \right)^2 + \frac{3}{2} w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w} & \text{P}_{IV} \\
\frac{d^2 w}{dz^2} &= \left( \frac{1}{2w} + \frac{1}{w-1} \right) \left( \frac{dw}{dz} \right)^2 - \frac{1}{w} \frac{dw}{dz} + \frac{(w-1)^2}{z^2} \left( \alpha w + \frac{\beta}{w} \right) \\
&\quad + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1} \\
\frac{d^2 w}{dz^2} &= \frac{1}{2} \left( \frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z} \right) \left( \frac{dw}{dz} \right)^2 - \left( \frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z} \right) \frac{dw}{dz} \\
&\quad + \frac{w(w-1)(w-z)}{z^2(z-1)^2} \left\{ \alpha + \frac{\beta z}{w^2} + \gamma(z-1) + \frac{\delta z(z-1)}{(w-1)^2} \right\} & \text{P}_{VI}
\end{align*}
\]

where \(\alpha, \beta, \gamma\) and \(\delta\) are arbitrary constants.
Outline

1. Rational solutions of $P_{II}$

2. Rational and algebraic solutions of $P_{III}$

3. Rational solutions of $P_{IV}$

4. Rational and algebraic solutions of $P_{V}$

5. Discussion and concluding remarks
Definition  A critical point is a point where multi-valuedness can occur.

- **Single-valued**
  \[
  w(z) = \frac{1}{z - z_0} \quad \text{pole}
  \]
  \[
  w(z) = \exp\left(\frac{1}{z - z_0}\right) \quad \text{essential singularity}
  \]

- **Multi-valued**
  \[
  w(z) = \sqrt{z - z_0} \quad \text{algebraic branch point}
  \]
  \[
  w(z) = \ln(z - z_0) \quad \text{logarithmic branch point}
  \]
  \[
  w(z) = \tan[\ln(z - z_0)] \quad \text{essential singularity}
  \]

Picard [1887] posed the problem of determining which ODEs have solutions with **no movable critical points**, or equivalently **no movable branch points**.

- This is now known as the “Painlevé property”.

- ODEs whose solutions possess it are said to be of “Painlevé type”.

- The **general solutions** of the Painlevé equations are **transcendental**, i.e. irreducible in the sense that they cannot be expressed in terms of previously known functions, such as rational functions or the classical special functions.
History of the Painlevé Equations

- Derived by Painlevé, Gambier and colleagues in the late 19th/early 20th centuries
- Studied in Minsk, Belarus by Erugin, Lukashevich, Gromak et al. since 1950’s; much of their work is published in the journal Diff. Eqns., the translation of Diff. Urav.
- Barouch, McCoy, Tracy & Wu [1973, 1976] showed that the correlation function of the two-dimensional Ising model is expressible in terms of solutions of P_{III}.
- Ablowitz and Segur [1977] demonstrated a close connection between completely integrable PDEs solvable by inverse scattering, the soliton equations, such as the Korteweg-de Vries and nonlinear Schrödinger equations, and the Painlevé equations.
- Flaschka and Newell [1980] introduced the isomonodromy deformation method (inverse scattering for ODEs), which expresses the Painlevé equation as the compatibility condition of two linear systems of equations and are studied using Riemann-Hilbert methods. Subsequent developments by Deift, Fokas, Its, Zhou, ...
- Algebraic and geometric studies of the Painlevé equations by Okamoto in 1980’s. Subsequent developments by Noumi, Umemura, Yamada, ...
- The Painlevé equations are due to form a chapter in the “Digital Library of Mathematical Functions”, which is a rewrite/update of Abramowitz & Stegun’s “Handbook of Mathematical Functions” (see http://dlmf.nist.gov).
Inverse Problems for the Painlevé Equations

Theorem (Ablowitz & Segur [1977])

Consider the integral equation

\[ K(z, \xi) = k \text{Ai} \left( \frac{z + \xi}{2} \right) + \frac{k^2}{4} \int_{z}^{\infty} \int_{z}^{\infty} K(z, s) \text{Ai} \left( \frac{s + t}{2} \right) \text{Ai} \left( \frac{t + \xi}{2} \right) ds dt \]

Then \( w(z) = K(z, z) \) satisfies

\[ w'' = 2w^3 + zw \]

which is the special case of \( P_{\Pi} \) with \( \alpha = 0 \), and the boundary condition

\[ w(z) \sim k \text{Ai}(z), \quad \text{as} \quad z \to \infty \]

Theorem (Flaschka & Newell [1980])

\[ w'' = 2w^3 + zw + \alpha \]

is the compatibility condition of the linear system

\[
\frac{\partial \Psi}{\partial z} = \begin{pmatrix} -i\lambda & w \\ w & i\lambda \end{pmatrix} \Psi, \quad \frac{\partial \Psi}{\partial \lambda} = \begin{pmatrix} -i(4\lambda^2 + 2w^2 + z) & 4\lambda w + 2iw' + \frac{\alpha}{\lambda} \\ 4\lambda w - 2iw' + \frac{\alpha}{\lambda} & i(4\lambda^2 + 2w^2 + z) \end{pmatrix} \Psi
\]

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Some Properties of the Painlevé Equations

- \( P_{\text{II}} - P_{\text{VI}} \) have **Bäcklund transformations** which map solutions of a given Painlevé equation to solutions of the same Painlevé equation, but with different values of the parameters, and have associated **Affine Weyl groups** which act on the parameter space.

- \( P_{\text{II}} - P_{\text{VI}} \) have **rational** and **algebraic solutions** for certain values of the parameters.

- \( P_{\text{II}} - P_{\text{VI}} \) have **special function solutions** expressed in terms of the classical special functions \( \text{Airy } \text{Ai}(z), \text{Bi}(z), \text{Bessel } J_\nu(z), Y_\nu(z), \text{parabolic cylinder } D_\nu(z), \text{Whittaker } M_{\kappa,\mu}(z), W_{\kappa,\mu}(z) \) and hypergeometric \( 2F_1(a, b; c; z) \), for certain values of the parameters.

- These rational, algebraic and special function solutions of \( P_{\text{II}} - P_{\text{VI}} \) can often be written in **determinantal form**.

- \( P_{\text{I}} - P_{\text{VI}} \) can be written as a (non-autonomous) **Hamiltonian system**.

- \( P_{\text{I}} - P_{\text{VI}} \) can be written in **bilinear form**.

- \( P_{\text{I}} - P_{\text{VI}} \) possess **Lax pairs** (isomonodromy problems).
Applications of the Painlevé Equations

• Asymptotics of nonlinear evolution equations
• Statistical Mechanics: correlation functions of the $XY$ model, Ising model
• Random matrix theory
• Length of longest increasing subsequences
• Distribution of zeros of the Riemann zeta function
• Quantum gravity and quantum field theory
• Topological field theory (WDVV equations)
• Solutions of the SDYM and stationary, axisymmetric Einstein equations
• Surfaces with constant negative curvature
• General relativity and plasma Physics
• Nonlinear waves: resonant oscillations in shallow water, convective flows with viscous dissipation, Görtler vortices in boundary layers, Hele-shaw problems
• Polyelectrolytes, Electrolysis and Superconductivity
• Bose-Einstein condensation
• Nonlinear optics and fibre optics
• Stimulated Raman Scattering
Solutions of $P_{II}$

$$
\frac{d^2w}{dz^2} = 2w^3 + zw + \alpha
$$

Theorem (Umemura & Watanabe [1997])

(1) For every $\alpha = N \in \mathbb{Z}$ there exists a unique rational solution of $P_{II}$, which have no arbitrary constants.

(2) For every $\alpha = N + \frac{1}{2}$, with $N \in \mathbb{Z}$, there exists a unique classical solution of $P_{II}$, each of which is rationally written in terms of Airy functions and has one arbitrary constant.

(3) For all other values of $\alpha$, the solution of $P_{II}$ is nonclassical, i.e. transcendental.

Remarks:

- General solutions of $P_{II}$, for all values of $\alpha$, are meromorphic, transcendental functions.

- Yablonskii & Vorob’ev [1965] proved (1), and

Rational Solutions of P\(_{II}\) — Vorob’ev–Yablonskii Polynomials

**Theorem** (Yablonskii & Vorob’ev [1965])

Suppose that \( Q_n(z) \) satisfies the recursion relation

\[
Q_{n+1}Q_{n-1} = zQ_n^2 - 4 \left[ Q_nQ''_n - (Q'_n)^2 \right]
\]

with \( Q_0(z) = 1 \) and \( Q_1(z) = z \). Then the rational function

\[
w(z; n) = \frac{d}{dz} \left\{ \ln \left[ \frac{Q_{n-1}(z)}{Q_n(z)} \right] \right\} = \frac{Q'_{n-1}(z)}{Q_{n-1}(z)} - \frac{Q'_n(z)}{Q_n(z)}
\]

satisfies \( P_{II} \)

\[
w'' = 2w^3 + zw + \alpha
\]

with \( \alpha = n \in \mathbb{Z}^+ \). Further \( w(z; 0) = 0 \) and \( w(z; -n) = -w(z; n) \).

The first few Yablonskii–Vorob’ev polynomials, which are monic polynomials of degree \( \frac{1}{2} n(n + 1) \), are

- \( Q_2(z) = z^3 + 4 \)
- \( Q_3(z) = z^6 + 20z^3 - 80 \)
- \( Q_4(z) = z^{10} + 60z^7 + 11200z \)
- \( Q_5(z) = z^{15} + 140z^{12} + 2800z^9 + 78400z^6 - 313600z^3 - 6272000 \)
- \( Q_6(z) = z^{21} + 280z^{18} + 18480z^{15} + 627200z^{12} - 17248000z^9 + 1448832000z^6 \\
+ 19317760000z^3 - 38635520000 \)
Plots of the Roots of the Yablonskii–Vorob’ev Polynomials $Q_n(z)$

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Roots of Yablonskii–Vorob’ev Polynomial $Q_{25}(z)$
Theorem (Fukutani, Okamoto & Umemura [2000])

• For every positive integer $n$, $Q_n(z)$ has simple roots, and $Q_n(z)$ and $Q_{n+1}(z)$ do not have a common root.

Theorem (Taneda [2000])

• $Q_n(z)$ is divisible by $z$ if and only if $n \equiv 1 \pmod{3}$.
• $Q_n(z)$ is a polynomial in $z^3$ if $n \not\equiv 1 \pmod{3}$ and $Q_n(z)/z$ is a polynomial in $z^3$ if $n \equiv 1 \pmod{3}$.

Remarks

• The distribution of the zeroes of the Yablonskii–Vorob’ev polynomials $Q_n(z)$ was investigated by Kametaka, Noda, Fukui & Hirano [1986] — see also Iwasaki, Kimura, Shimomura & Yoshida [1991].
• The hierarchy of rational solutions can be derived using the Bäcklund transformation

$$w_{n+1} = -w_n - \frac{2n + 1}{2w_n^2 + 2w'_n + z}$$

where $w_m = w(z; m)$, with “seed solution” $w_0 = 0$. 
• Each polynomial \( Q_n(z) \) has only simple roots so

\[
Q_n(z) = \prod_{k=1}^{n(n+1)/2} (z - a_{n,k})
\]

where \( a_{n,k} \), for \( k = 1, 2, \ldots, \frac{1}{2}n(n + 1) \), are the roots. Thus rational solutions of \( P_{II} \) have the form

\[
w(z; n) = \frac{Q'_{n-1}(z)}{Q_{n-1}(z)} - \frac{Q'_n(z)}{Q_n(z)} = \sum_{k=1}^{n(n-1)/2} \frac{1}{z - a_{n-1,k}} - \sum_{k=1}^{n(n+1)/2} \frac{1}{z - a_{n,k}}
\]

and so \( w(z; n) \) has \( n \) roots, \( \frac{1}{2}n(n-1) \) with residue \(+1\) and \( \frac{1}{2}n(n+1) \) with residue \(-1\).

• The roots \( a_{n,k} \) of \( Q_n(z) \) satisfy

\[
\sum_{k=1, k\neq j}^{n(n+1)/2} \frac{1}{(a_{n,j} - a_{n,k})^3} = 0, \quad j = 1, 2, \ldots, \frac{1}{2}n(n + 1)
\]

This follows from the study of rational solutions of the Korteweg-de Vries (KdV) equation

\[
 u_t + 6uu_x + u_{xx} = 0
\]

and a related many-body problem by Airault, McKean and Moser [1977].
Determinantal Form of Rational Solutions of $P_{II}$

**Theorem**

(Kajiwara & Ohta [1996])

Let $\varphi_k(z)$ be the polynomial defined by

$$\sum_{k=0}^{\infty} \varphi_k(z) \lambda^k = \exp \left( z \lambda - \frac{4}{3} \lambda^3 \right)$$

and $\tau_n(z)$ be the $n \times n$ determinant given by the Wronskian

$$\tau_n(z) = \mathcal{W}(\varphi_1, \varphi_3, \ldots, \varphi_{2n-1}) \equiv \begin{vmatrix} \varphi_1 & \varphi_3 & \cdots & \varphi_{2n-1} \\ \varphi'_1 & \varphi'_3 & \cdots & \varphi'_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{(n-1)} & \varphi_{(n-1)} & \cdots & \varphi_{(n-1)} \end{vmatrix}$$

then

$$w_n(z) = \frac{d}{dz} \ln \left( \frac{\tau_{n-1}(z)}{\tau_n(z)} \right)$$

satisfies $P_{II}$ with $\alpha = n$.

- Flaschka and Newell [1980], following the earlier work of Airault [1979], expressed the rational solutions of $P_{II}$ as the logarithmic derivatives of determinants.
- Note that

$$\tau_n(z) = c_n Q_n(z), \quad c_n = \prod_{j=1}^{n} (2j + 1)^{j-n}$$
Rational Solutions of the $P_{II}$ Hamiltonian

$P_{II}$ can be written as the Hamiltonian system

$$\frac{dq}{dz} = \frac{\partial H_{II}}{\partial p} = p - q^2 - \frac{1}{2} z,$$
$$\frac{dp}{dz} = - \frac{\partial H_{II}}{\partial q} = 2qp + \alpha + \frac{1}{2}$$

(1)

where the (non-autonomous) Hamiltonian $H_{II}(q, p, z; \alpha)$ is given by

$$H_{II}(q, p, z; \alpha) = \frac{1}{2} p^2 - (q^2 + \frac{1}{2} z)p - (\alpha + \frac{1}{2})q$$

The Hamiltonian function $\sigma = H_{II}(q, p, z; \alpha)$ satisfies (Jimbo and Miwa [1981], Okamoto [1986])

$$(\sigma'')^2 + 4(\sigma')^3 + 2\sigma'(z\sigma' - \sigma) = \frac{1}{4}(\alpha + \frac{1}{2})^2$$

(2)

Due to the relationship between the Hamiltonian function and associated $\tau$-functions it can be shown that rational solutions of (2) have the form

$$\sigma_n = -\frac{1}{8} z^2 + (\ln Q_n)'$$

(3)

Differentiating (2) with respect to $z$ yields

$$\sigma'''' + 6 (\sigma')^2 + 2z\sigma' - \sigma = 0$$

and then substituting (3) into this yields

$$Q_n Q_n''' - 4Q_n' Q_n''' + 3 (Q_n'')^2 - z \left[ Q_n Q_n'' - (Q_n')^2 \right] - Q_n Q_n' = 0$$
The Yablonskii–Vorob’ev polynomials $Q_n(z)$, which are defined the second order, bilinear differential-difference equation

$$Q_{n+1}Q_{n-1} = zQ_n^2 - 4 \left[ Q_nQ''_n - (Q'_n)^2 \right]$$

also satisfy the fourth order, bilinear equation

$$Q_nQ''''_n - 4Q'_nQ'''_n + 3(Q''_n)^2 - z \left[ Q_nQ''_n - (Q'_n)^2 \right] - Q_nQ'_n = 0$$

and the fifth order, quad-linear (i.e. homogeneous of degree 4) difference equation

$$\frac{Q_{n+2}Q_{n-1}^2Q_{n-2}}{2n+1} - \frac{Q_{n+1}Q_n^2Q_{n-3}}{2n-1} + \frac{2Q_{n+1}^2Q_{n-2}^2}{4n^2-1} + 4(2n+1)Q_n^3Q_{n-2} - 4(2n-1)Q_{n+1}Q_{n-1}^3 = 0$$

Additionally $Q_n$ satisfies the differential-difference equations

$$Q''_{n+1}Q_n - 2Q'_{n+1}Q'_n + Q_{n+1}Q''_n = 0$$

$$Q''''_{n+1}Q_n - 3Q''_{n+1}Q'_n + 3Q'_{n+1}Q''_n - Q_{n+1}Q'''_n$$
$$- z(Q'_{n+1}Q_n - Q_{n+1}Q'_n) + (n + 1)Q_{n+1}Q_n = 0$$

$$Q'_{n+1}Q_{n-1} - Q_{n+1}Q'_{n-1} = (2n + 1)Q_n^2$$
Rational Solutions of $P_{III}$

Consider the generic case of $P_{III}$ when $\gamma \delta \neq 0$, then we set $\gamma = 1$ and $\delta = -1$, without loss of generality (by rescaling $w$ and $z$ if necessary), and so

$$
\frac{d^2 w}{dz^2} = \frac{1}{w} \left( \frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{\alpha w^2 + \beta}{z} + w^3 - \frac{1}{w}
$$

**Theorem**  
(Lukashevich [1967], Murata [1995], Milne, PAC & Bassom [1997])  

$P_{III}$ with $\gamma = -\delta = 1$, has rational solutions if and only if

$$
\varepsilon_1 \alpha + \varepsilon_2 \beta = 4n
$$

with $n \in \mathbb{Z}$ and $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$, independently. Generically (except when $\alpha$ and $\beta$ are both integers) these rational solutions have the form

$$
w(z) = P_{n^2}(z)/Q_{n^2}(z)
$$

where $P_{n^2}(z)$ and $Q_{n^2}(z)$ are polynomials of degree $n^2$ with no common roots.
Theorem (Kajiwara & Masuda [1999], PAC [2003])

Suppose that $S_n(z; \mu)$ satisfies the recursion relation

$$S_{n+1}S_{n-1} = -z \left[ S_nS''_n - (S'_n)^2 \right] - S_nS'_n + (z + \mu)S^2_n$$

with $S_{-1}(z; \mu) = S_0(z; \mu) = 1$. Then

$$w_n = w(z; \alpha_n, \beta_n, 1, -1) = 1 + \frac{d}{dz} \left\{ \ln \left[ \frac{S_{n-1}(z; \mu - 1)}{S_n(z; \mu)} \right] \right\} \equiv \frac{S_n(z; \mu - 1) S_{n-1}(z; \mu)}{S_n(z; \mu) S_{n-1}(z; \mu - 1)}$$

satisfies $P_{III}$

$$w''_n = \frac{(w'_n)^2}{w_n} - \frac{w'_n}{z} + \frac{\alpha_n w^2_n + \beta_n}{z} + w^3_n - \frac{1}{w_n}$$

with $\alpha_n = 2n + 2\mu - 1$ and $\beta_n = 2n - 2\mu + 1$.

The first few polynomials, which are monic polynomials of degree $\frac{1}{2}n(n+1)$ with integer coefficients, are

- $S_1(z; \mu) = \xi$
- $S_2(z; \mu) = \xi^3 - \mu$
- $S_3(z; \mu) = \xi^6 - 5\mu \xi^3 + 9\mu \xi - 5\mu^2$
- $S_4(z; \mu) = \xi^{10} - 15\mu \xi^7 + 63\mu \xi^5 - 225\mu \xi^3 + 315\mu^2 \xi^2 - 175\mu^3 \xi + 36\mu^2$

with $\xi = z - \mu$. 

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**Discriminants**

Let \( f(z) = z^m + a_{m-1}z^{m-1} + \ldots + a_1z + a_0 \) be a monic polynomial of degree \( m \) with roots \( \alpha_1, \alpha_2, \ldots, \alpha_m \), so

\[
f(z) = \prod_{j=1}^{m} (z - \alpha_j)
\]

**Definition**

- The **discriminant** of \( f(z) \) is \( \text{Dis}(f) = \prod_{1 \leq j < k \leq m} (\alpha_j - \alpha_k)^2 \). Thus \( \text{Dis}(f) = 0 \) if and only if \( f(z) \) has a multiple root.

\[
\begin{align*}
\text{Dis}(S_2(z; \mu)) & = -3^3\mu^2 \\
\text{Dis}(S_3(z; \mu)) & = 3^{12}5^5\mu^6(\mu^2 - 1)^2 \\
\text{Dis}(S_4(z; \mu)) & = 3^{27}5^{20}7^7\mu^{14}(\mu^2 - 1)^6(\mu^2 - 4)^2 \\
\text{Dis}(S_5(z; \mu)) & = 3^{66}5^{45}7^{28}\mu^{26}(\mu^2 - 1)^{14}(\mu^2 - 4)^6(\mu^2 - 9)^2 \\
\text{Dis}(S_6(z; \mu)) & = -3^{147}5^{80}7^{63}11^{11}\mu^{44}(\mu^2 - 1)^{26}(\mu^2 - 4)^{14}(\mu^2 - 9)^6(\mu - 16)^2
\end{align*}
\]

**Conjecture**

- \( \text{Dis}(S_n(z; \mu)) = 0 \) when \( \mu = 0, \pm 1, \pm 2, \ldots, \pm (n - 2) \). Further these multiple roots of \( S_n(z; \mu) \) occur at \( z = 0 \), which is a singular point of \( \Pi_3 \).
Resultants

Let \( f(z) = z^m + a_{m-1}z^{m-1} + \ldots + a_1z + a_0 \) and \( g(z) = z^n + b_{n-1}z^{n-1} + \ldots + b_1z + b_0 \) be monic polynomials of degree \( m \) and \( n \) with roots \( \alpha_1, \alpha_2, \ldots, \alpha_m \) and \( \beta_1, \beta_2, \ldots, \beta_n \), so

\[
  f(z) = \prod_{j=1}^{m} (z - \alpha_j), \quad g(z) = \prod_{k=1}^{n} (z - \beta_k)
\]

Definition

- The resultant of \( f(z) \) and \( g(z) \) is \( \text{Res}(f, g) = \prod_{k=1}^{n} \prod_{j=1}^{m} (\alpha_j - \beta_k) \). Thus \( \text{Res}(f, g) = 0 \) if and only if \( f(z) \) and \( g(z) \) have a common root.

\[
  \begin{align*}
    \text{Res}(S_1, S_2) &= -\mu \\
    \text{Res}(S_2, S_3) &= -3^6 \mu^4 (\mu^2 - 1) \\
    \text{Res}(S_3, S_4) &= -3^{18} 5^{10} \mu^{10} (\mu^2 - 1)^4 (\mu^2 - 4) \\
    \text{Res}(S_4, S_5) &= -3^{36} 5^{30} 7^{14} (\mu^2 - 1)^{10} (\mu^2 - 4)^4 (\mu^2 - 9) \\
    \text{Res}(S_5, S_6) &= 3^{96} 5^{60} 7^{42} \mu^{35} (\mu^2 - 1)^{20} (\mu^2 - 4)^{10} (\mu^2 - 9)^4 (\mu - 16)
  \end{align*}
\]

Conjecture

- \( \text{Res}(S_n, S_{n+1}) = 0 \) when \( \mu = 0, \pm 1, \pm 2, \ldots, \pm (n - 1) \).
\[ S_3(z; \mu) = z^6 + 6\mu z^5 + 15\mu^2 z^4 + 5\mu(4\mu^2 - 1)z^3 + 15\mu^2(\mu^2 - 1)z^2 \\
\quad + 3\mu(\mu^2 - 1)(2\mu^2 - 3)z + \mu^2(\mu^2 - 1)(\mu^2 - 4) \]

\[ S_3(z; 0) = z^6 \]
\[ S_3(z; 1) = (z^3 + 6z^2 + 15z + 15)z^3 \]
\[ S_3(z; 2) = (z^5 + 12z^4 + 60z^3 + 150z^2 + 180z + 90)z \]

\[ S_4(z; \mu) = z^{10} + 10\mu z^9 + 45\mu^2 z^8 + 15\mu(8\mu^2 - 1)z^7 + 105\mu^2(2\mu^2 - 1)z^6 \\
\quad + 63\mu(4\mu^2 - 1)(\mu^2 - 1)z^5 + 105\mu^2(\mu^2 - 1)(2\mu^2 - 3)z^4 \\
\quad + 15\mu(\mu^2 - 1)(8\mu^4 - 27\mu^2 + 15)z^3 + 45\mu^2(\mu^2 - 1)(\mu^2 - 2)(\mu^2 - 4)z^2 \\
\quad + 5\mu^3(\mu^2 - 1)(\mu^2 - 4)(2\mu^2 - 11)z + \mu^2(\mu^2 - 1)^2(\mu^2 - 4)(\mu^2 - 9) \]

\[ S_4(z; 0) = z^{10} \]
\[ S_4(z; 1) = (z^4 + 10z^3 + 45z^2 + 105z + 105)z^6 \]
\[ S_4(z; 2) = (z^7 + 20z^6 + 180z^5 + 930z^4 + 2940z^3 + 5670z^2 + 6300z + 3150)z^3 \]
\[ S_4(z; 3) = (z^9 + 30z^8 + 405z^7 + 3195z^6 + 16065z^5 + 52920z^4 \\
\quad + 113400z^3 + 151200z^2 + 113400z + 37800)z \]
Algebraic Solutions of $P_{III}$

Consider the special case of $P_{III}$ when either (i), $\gamma = 0$ and $\alpha \delta \neq 0$, or (ii), $\delta = 0$ and $\beta \gamma \neq 0$. In case (i), we make the transformation

$$w(z) = \left(\frac{2}{3}\right)^{1/2}u(\zeta), \quad z = \left(\frac{2}{3}\right)^{3/2}\zeta^3$$

and set $\alpha = 1$, $\beta = 2\mu$ and $\delta = -1$, with $\mu$ an arbitrary constant, without loss of generality, which yields

$$\frac{d^2u}{d\zeta^2} = \frac{1}{u} \left(\frac{du}{d\zeta}\right)^2 - \frac{1}{\zeta} \frac{du}{d\zeta} + 4\zeta u^2 + 12\mu \zeta - \frac{4\zeta^4}{u}$$

$P_{III}^{(7)}$

In case (ii), we make the transformation

$$w(z) = \left(\frac{3}{2}\right)^{1/2}/u(\zeta), \quad z = \left(\frac{2}{3}\right)^{3/2}\zeta^3$$

and set $\alpha = 2\mu$, $\beta = -1$ and $\gamma = 1$, with $\mu$ an arbitrary constant, without loss of generality, which again yields $P_{III}^{(7)}$. 

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\[
\frac{d^2 u}{d\zeta^2} = \frac{1}{u} \left( \frac{du}{d\zeta} \right)^2 - \frac{1}{\zeta} \frac{du}{d\zeta} + 4\zeta u^2 + 12\mu \zeta - \frac{4\zeta^4}{u}
\]

**Theorem**

\(P_{\text{III}}^{(7)}\) has rational solutions if and only if \(\mu = n\), with \(n \in \mathbb{Z}\). These rational solutions have the form

\[
u_n(\zeta) = \frac{P_{n^2+1}(\zeta)}{Q_{n^2}(\zeta)}
\]

where \(P_m(\zeta)\) and \(Q_m(\zeta)\) are monic polynomials of degree \(m\) with integer coefficients and no common roots.

**Theorem**

(Ohyama [2001], PAC [2003])

Suppose that \(R_n(\zeta)\) satisfies the recursion relation

\[
2\zeta R_{n+1}R_{n-1} = -R_n \frac{d^2 R_n}{d\zeta^2} + \left( \frac{dR_n}{d\zeta} \right)^2 - \frac{R_n}{\zeta} \frac{dR_n}{d\zeta} + 2(\zeta^2 - n)R_n^2
\]

with \(R_0(\zeta) = 1\) and \(R_1(\zeta) = \zeta^2\). Then

\[
u_n(\zeta) = \frac{R_{n+1}(\zeta) R_{n-1}(\zeta)}{R_n^2(\zeta)}
\]

satisfies \(P_{\text{III}}^{(7)}\) with \(\mu = n\). Additionally \(u_{-n}(\zeta) = -iu_n(i\zeta)\).
• The first few polynomials $R_n(\zeta)$ are

\[
\begin{align*}
R_2(\zeta) &= (\zeta^2 - 1)\zeta^3 \\
R_3(\zeta) &= (\zeta^4 - 4\zeta^2 + 5)\zeta^5 \\
R_4(\zeta) &= (\zeta^8 - 10\zeta^6 + 40\zeta^4 - 70\zeta^2 + 35)\zeta^6 \\
R_5(\zeta) &= (\zeta^{12} - 20\zeta^{10} + 175\zeta^8 - 840\zeta^6 + 2275\zeta^4 - 3220\zeta^2 + 1925)\zeta^8 \\
R_6(\zeta) &= (\zeta^{18} - 35\zeta^{16} + 560\zeta^{14} - 5320\zeta^{12} - 32690\zeta^{10} + 133070\zeta^8 - 354200\zeta^6 \\
&\quad + 585200\zeta^4 - 525525\zeta^2 + 175175)\zeta^9
\end{align*}
\]

• Associated rational solutions of $P_{\text{III}}^{(7)}$ are

\[
\begin{align*}
\eta_1(\zeta) &= \frac{\zeta^2 - 1}{\zeta} \\
\eta_2(\zeta) &= \frac{\zeta(\zeta^4 - 4\zeta^2 + 5)}{(\zeta^2 - 1)^2} \\
\eta_3(\zeta) &= \frac{(\zeta^2 - 1)(\zeta^8 - 10\zeta^6 + 40\zeta^4 - 70\zeta^2 + 35)}{\zeta(\zeta^4 - 4\zeta^2 + 5)^2} \\
\eta_4(\zeta) &= \frac{\zeta(\zeta^4 - 4\zeta^2 + 5)(\zeta^{12} - 20\zeta^{10} + 175\zeta^8 - 840\zeta^6 + 2275\zeta^4 - 3220\zeta^2 + 1925)}{(\zeta^8 - 10\zeta^6 + 40\zeta^4 - 70\zeta^2 + 35)^2}.
\end{align*}
\]
Poles of Rational Solutions $u_n(\zeta)$ of $P_{\text{III}}^{(7)}$
Poles of $u_{20}(\zeta)$
Rational Solution Hierarchies of $P_{IV}$

$$\frac{d^2w}{dz^2} = \frac{1}{2w} \left( \frac{dw}{dz} \right)^2 + \frac{3}{2} w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w} \quad P_{IV}$$

**Theorem** (Lukashevich [1967], Gromak [1987], Murata [1985])

$P_{IV}$ has rational solutions if and only if

(i) $\alpha = m$, $\beta = -2(2n - m + 1)^2$ or (ii) $\alpha = m$, $\beta = -2(2n - m + \frac{1}{3})^2$

with $m, n \in \mathbb{Z}$. Further the rational solutions for these parameter values are unique.

Simple rational solutions $P_{IV}$ are

$$w(z; \pm 2, -2) = \pm 1/z, \quad w(z; 0, -2) = -2z, \quad w(z; 0, -\frac{2}{9}) = -\frac{2}{3}z$$

There are three hierarchies of rational solutions derived using Bäcklund transformations

$$w(z; \alpha_1, \beta_1) = P_{1,n-1}(z)/Q_{1,n}(z) \quad -1/z \text{ hierarchy}$$

$$w(z; \alpha_2, \beta_2) = -2z + P_{2,n-1}(z)/Q_{2,n}(z) \quad -2z \text{ hierarchy}$$

$$w(z; \alpha_3, \beta_3) = -\frac{2}{3}z + P_{3,n-1}(z)/Q_{3,n}(z) \quad -\frac{2}{3}z \text{ hierarchy}$$

where $P_{j,n}(z)$ and $Q_{j,n}(z)$ are polynomials of degree $n$ with no common roots. The “$-1/z$” and “$-2z$” hierarchies generate the rational solutions in case (i) and the “$-\frac{2}{3}z$” hierarchy generates the rational solutions in case (ii).
$P_{IV}$ Rational Solutions

$a = \alpha, \quad b = \sqrt{-2/\beta}$
Symmetric Form of $P_{IV}$

(Bureau [1980], Adler [1994], Noumi & Yamada [1998])

Consider the symmetric $P_{IV}$ system

\[
\begin{align*}
\varphi_0' + \varphi_0(\varphi_1 - \varphi_2) + 2\mu_0 &= 0 \\
\varphi_1' + \varphi_1(\varphi_2 - \varphi_0) + 2\mu_1 &= 0 \\
\varphi_2' + \varphi_2(\varphi_0 - \varphi_1) + 2\mu_2 &= 0
\end{align*}
\]

where $\mu_0$, $\mu_1$ and $\mu_2$ are constants, $\varphi_0$, $\varphi_1$ and $\varphi_2$ are functions of $z$, with

\[
\begin{align*}
\mu_0 + \mu_1 + \mu_2 &= 1 \\
\varphi_0 + \varphi_1 + \varphi_2 + 2z &= 0
\end{align*}
\]

Eliminating $\varphi_1$ and $\varphi_2$, then $w = \varphi_0$ satisfies $P_{IV}$

\[
w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}
\]

with

\[
\alpha = \mu_0 - \mu_2, \quad \beta = -2\mu_0^2
\]

The system (1) is associated with the affine Weyl group $A_2^{(1)}$. 
Suppose that \( H_{m,n}(z) \), with \( m, n \geq 0 \), satisfies the recurrence relations
\[
2mH_{m+1,n}H_{m-1,n} = H_{m,n}H''_{m,n} - (H'_{m,n})^2 + 2mH^2_{m,n} \\
-2nH_{m,n+1}H_{m,n-1} = H_{m,n}H''_{m,n} - (H'_{m,n})^2 - 2nH^2_{m,n}
\]
with
\[
H_{0,0} = H_{1,0} = H_{0,1} = 1, \quad H_{1,1} = 2z
\]
then
\[
w^{(I)}_{m,n} = w(z; \alpha^{(I)}_{m,n}, \beta^{(I)}_{m,n}) = -\frac{d}{dz} \ln \left( \frac{H_{m,n+1}}{H_{m,n}} \right) \equiv -2m \frac{H_{m+1,n} H_{m-1,n+1}}{H_{m,n+1} H_{m,n}}
\]
\[
w^{(II)}_{m,n} = w(z; \alpha^{(II)}_{m,n}, \beta^{(II)}_{m,n}) = \frac{d}{dz} \ln \left( \frac{H_{m+1,n}}{H_{m,n}} \right) \equiv 2n \frac{H_{m,n+1} H_{m+1,n-1}}{H_{m+1,n} H_{m,n}}
\]
are respectively solutions of \( P_{IV} \)
\[
w'' = \left(\frac{w'}{2w}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}
\]
for
\[
\alpha^{(I)}_{m,n} = -m - 2n - 1, \quad \beta^{(I)}_{m,n} = -2m^2 \\
\alpha^{(II)}_{m,n} = 2m + n + 1, \quad \beta^{(II)}_{m,n} = -2n^2
\]
Roots of the Generalized Hermite Polynomials $H_{m,n}(z)$
Roots of $H_{20,20}(z)$

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Theorem (Noumi & Yamada [1998])

Suppose that $Q_{m,n}(z)$, with $m, n \in \mathbb{Z}$, satisfies the recurrence relations

\[
Q_{m+1,n}Q_{m-1,n} = \frac{9}{2} \left[ Q_{m,n}^{(m,n)} - (Q_{m,n}')^2 \right] + \left[ 2z^2 + 3(2m + n - 1) \right] Q_{m,n}^2
\]

\[
Q_{m,n+1}Q_{m,n-1} = \frac{9}{2} \left[ Q_{m,n}^{(m,n)} - (Q_{m,n}')^2 \right] + \left[ 2z^2 + 3(1 - m - 2n) \right] Q_{m,n}^2
\]

with $Q_{0,0} = Q_{1,0} = Q_{0,1} = 1$ and $Q_{1,1} = \sqrt{2} z$ then

\[
\tilde{w}^{(i)}_{m,n} = w(z; \tilde{\alpha}_{m,n}^{(i)}, \tilde{\beta}_{m,n}^{(i)}) = -\frac{2}{3}z + \frac{d}{dz} \ln \left( \frac{Q_{m+1,n}}{Q_{m,n}} \right) \equiv -\frac{1}{3} \sqrt{2} \frac{Q_{m+1,n} Q_{m-1,n+1}}{Q_{m,n} Q_{m,n+1}}
\]

\[
\tilde{w}^{(ii)}_{m,n} = w(z; \tilde{\alpha}_{m,n}^{(ii)}, \tilde{\beta}_{m,n}^{(ii)}) = -\frac{2}{3}z + \frac{d}{dz} \ln \left( \frac{Q_{m+1,n}}{Q_{m,n}} \right) \equiv -\frac{1}{3} \sqrt{2} \frac{Q_{m,n+1} Q_{m+1,n-1}}{Q_{m,n} Q_{m+1,n}}
\]

are respectively solutions of $P_{IV}$

\[
w'' = \left( \frac{w'}{2w} \right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}
\]

for

\[
\tilde{\alpha}_{m,n}^{(i)} = -m - 2n, \quad \tilde{\beta}_{m,n}^{(i)} = -\frac{2}{9}(3m - 1)^2
\]

\[
\tilde{\alpha}_{m,n}^{(ii)} = 2m + n, \quad \tilde{\beta}_{m,n}^{(ii)} = -\frac{2}{9}(3n - 1)^2
\]
Roots of the Generalized Okamoto Polynomials $Q_{m,n}(z)$
Roots of $Q_{10,10}(z)$
Rational Solutions of $P_V$

Theorem (Kitaev, Law & McLeod [1994])

$P_V$, with $\delta \neq 0$,

$$
    w'' = \left( \frac{1}{2w} + \frac{1}{w-1} \right) (w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2} \left( \alpha w + \frac{\beta}{w} \right) + \gamma \frac{w}{z} - \frac{w(w+1)}{2(w-1)}
$$

has a rational solution if and only if one of the following holds with $m, n \in \mathbb{Z}$ and $\varepsilon = \pm 1$.

(i), $\alpha = \frac{1}{2}(m + \varepsilon)^2$ and $\beta = -\frac{1}{2}n^2$, where $n > 0$, $m + n$ is odd, and $\alpha \neq 0$ when $|m| < n$,

(ii), $\alpha = \frac{1}{2}n^2$ and $\beta = -\frac{1}{2}(m + \varepsilon)^2$, where $n > 0$, $m + n$ is odd, and $\beta \neq 0$ when $|m| < n$,

(iii), $\alpha = \frac{1}{2}a^2$, $\beta = -\frac{1}{2}(a + n)^2$ and $\gamma = m$, where $m + n$ is even and $a$ arbitrary,

(iv), $\alpha = \frac{1}{2}(b + n)^2$, $\beta = -\frac{1}{2}b^2$ and $\gamma = m$, where $m + n$ is even and $b$ arbitrary,

(v), $\alpha = \frac{1}{8}(2m + 1)^2$ and $\beta = -\frac{1}{8}(2n + 1)^2$.

These rational solutions have the form

$$
    w(z) = \lambda z + \mu + \frac{P_{n-1}(z)}{Q_n(z)}
$$

where $\lambda, \mu$ are constants, and $P_m(z), Q_m(z)$ are polynomials of degree $m$ with no common roots.
Remarks

• The rational solutions in cases (i) and (ii) are the special cases of the solutions of $P_V$ expressible in terms of confluent hypergeometric functions $\,_{1}F_{1}(a; c; z)$, often called the *special function family of solutions*, which depend on one arbitrary constant, when the confluent hypergeometric function reduces to the associated Laguerre polynomial $L_k^{(m)}(\zeta)$; see Masuda [2003] for details of special function solutions of $P_V$.

• The rational solutions in cases (i) and (ii) and those in cases (iii) and (iv) are related by the symmetry $S_2$

\[
S_2 : \quad w_2(z) = 1/w(z) , \quad (\alpha_2, \beta_2, \gamma_2, \delta_2) = (-\beta, -\alpha, -\gamma, \delta)
\]

• Kitaev, Law & McLeod [1994] did not explicitly give case (iv), though it is obvious by applying the symmetry $S_2$ to case (iii).

• Kitaev, Law & McLeod [1994] also require that $\gamma \notin \mathbb{Z}$ in case (v), though this appears not to be necessary.
P_V — Generalized Umemura Polynomials

Theorem (Masuda, Ohta & Kajiwara [2001])

Suppose that $U_{m,n}(z; \mu)$ satisfies the recursion relations

\[
U_{m+1,n}U_{m-1,n} = 8z\left[U_{m,n}U''_{m,n} - \left(U'_m\right)^2\right] + 8U_{m,n}U'_m + (z + 2\mu - 2 - 6m + 2n)U_{m,n}^2
\]

\[
U_{m,n+1}U_{m,n-1} = 8z\left[U_{m,n}U''_{m,n} - \left(U'_m\right)^2\right] + 8U_{m,n}U'_m + (z - 2\mu - 2 + 2m - 6n)U_{m,n}^2
\]

with

\[
U_{-1,-1}(z; \mu) = U_{-1,0}(z; \mu) = U_{0,-1}(z; \mu) = U_{0,0}(z; \mu) = 1
\]

and $\mu$ an arbitrary constant. Then

\[
w_{m,n}^{(iii)}(z; \mu) = w(z; \alpha_{m,n}^{(iii)}, \beta_{m,n}^{(iii)}, \gamma_{m,n}^{(iii)}, \delta_{m,n}^{(iii)}) = -\frac{U_{m,n-1}(z; \mu)U_{m-1,n}(z; \mu)}{U_{m-1,n}(z; \mu - 2)U_{m,n-1}(z; \mu + 2)}
\]

is a rational solution of $P_V$ for

\[
(\alpha_{m,n}^{(iii)}, \beta_{m,n}^{(iii)}, \gamma_{m,n}^{(iii)}, \delta_{m,n}^{(iii)}) = \left(\frac{1}{8}\mu^2, -\frac{1}{8}(\mu - 2m + 2n)^2, -m - n, -\frac{1}{2}\right)
\]

and

\[
w_{m,n}^{(v)}(z; \mu) = w(z; \alpha_{m,n}^{(v)}, \beta_{m,n}^{(v)}, \gamma_{m,n}^{(v)}, \delta_{m,n}^{(v)}) = -\frac{U_{m,n-1}(z; \mu + 1)U_{m,n+1}(z; \mu - 1)}{U_{m-1,n}(z; \mu - 1)U_{m+1,n}(z; \mu + 1)}
\]

is a rational solution of $P_V$ for

\[
(\alpha_{m,n}^{(v)}, \beta_{m,n}^{(v)}, \gamma_{m,n}^{(v)}, \delta_{m,n}^{(v)}) = \left(\frac{1}{8}(2m + 1)^2, -\frac{1}{8}(2n + 1)^2, m - n - \mu, -\frac{1}{2}\right)
\]
Discriminants of Generalized Umemura Polynomials $U_{m,n}(z; \mu)$

\[
\begin{align*}
\text{Dis}(U_{1,1}(z; \mu)) &= 2^4(\theta^2 - 1) \\
\text{Dis}(U_{2,1}(z; \mu)) &= -2^{20}3^3\theta^4(\theta^2 - 4)^2 \\
\text{Dis}(U_{2,2}(z; \mu)) &= 2^{56}3^6(\theta^2 - 1)^8(\theta^2 - 9)^3 \\
\text{Dis}(U_{3,1}(z; \mu)) &= 2^{64}3^{12}5^5(\theta^2 - 1)^8(\theta^2 - 9)^3 \\
\text{Dis}(U_{3,2}(z; \mu)) &= -2^{120}3^{15}5^5\theta^{16}(\theta^2 - 4)^{12}(\theta^2 - 16)^4 \\
\text{Dis}(U_{3,3}(z; \mu)) &= 2^{224}3^{24}5^{10}(\theta^2 - 1)^{25}(\theta^2 - 9)^{16}(\theta^2 - 25)^5 \\
\text{Dis}(U_{4,1}(z; \mu)) &= 2^{160}3^{37}5^{20}7^7\theta^{16}(\theta^2 - 1)(\theta^2 - 4)^{12}(\theta^2 - 16)^5 \\
\text{Dis}(U_{4,2}(z; \mu)) &= -2^{248}3^{30}5^{20}7^7(\theta^2 - 1)^{25}(\theta^2 - 9)^{16}(\theta^2 - 25)^5 \\
\text{Dis}(U_{4,3}(z; \mu)) &= 2^{400}3^{39}5^{25}7^7\theta^{40}(\theta^2 - 4)^{34}(\theta^2 - 16)^{20}(\theta^2 - 36)^6 \\
\text{Dis}(U_{4,4}(z; \mu)) &= 2^{640}3^{54}5^{40}7^{14}(\theta^2 - 1)^{56}(\theta^2 - 9)^{43}(\theta^2 - 25)^{24}(\theta^2 - 49)^5
\end{align*}
\]

where $\theta = \mu - m + n$
Algebraic Solutions of \( P_V \)

If in \( P_V \), \( \delta = 0 \) and \( \gamma \neq 0 \), then it is equivalent to the generic case if \( P_{III} \).

**Theorem** (Gromak [1975])

Suppose that \( v = v(\zeta; a, b, 1, -1) \) is a solution of \( P_{III} \)

\[
\frac{d^2v}{d\zeta^2} = \frac{1}{v} \left( \frac{dv}{d\zeta} \right)^2 - \frac{1}{\zeta} \frac{dv}{d\zeta} + \frac{av^2 + b}{\zeta} + cv^3 + \frac{d}{v}
\]

and

\[
\eta(\zeta) = \frac{dv}{d\zeta} - \varepsilon v^2 + \frac{(1 - \varepsilon a)v}{\zeta}
\]

with \( \varepsilon^2 = 1 \). Then

\[
w(z; \alpha, \beta, \gamma, \delta) = \frac{\eta(\zeta) - 1}{\eta(\zeta) + 1}, \quad z = \frac{1}{2} \zeta^2
\]

satisfies \( P_V \) with

\[
(\alpha, \beta, \gamma, \delta) = ((b - \varepsilon a + 2)^2/32, -(b + \varepsilon a - 2)^2/32, -\varepsilon, 0)
\]

Making the change of variables \( w(z) = u(\zeta) \), with \( z = \frac{1}{2} \zeta^2 \), in \( P_V \) with \( \delta = 0 \) yields

\[
\frac{d^2u}{d\zeta^2} = \left( \frac{1}{2u} + \frac{1}{u - 1} \right) \left( \frac{du}{d\zeta} \right)^2 - \frac{1}{\zeta} \frac{du}{d\zeta} + \frac{4(u - 1)^2}{\zeta^2} \left( \alpha u + \frac{\beta}{u} \right) + 2\gamma u \quad (1)
\]
Algebraic solutions of $P_V$ with $\delta = 0$ and $\gamma \neq 0$ are equivalent to rational solutions of
\[
\frac{d^2 u}{d\zeta^2} = \left( \frac{1}{2u} + \frac{1}{u-1} \right) \left( \frac{du}{d\zeta} \right)^2 - \frac{1}{\zeta} \frac{du}{d\zeta} + \frac{4(u-1)^2}{\zeta^2} \left( \alpha u + \frac{\beta}{u} \right) + 2\gamma u
\]
which are classified as follows.

**Theorem** (Murata [1995], Milne, PAC & Bassom [1997])

Necessary and sufficient conditions for the existence of rational solutions of (1) are either
\[
(\alpha, \beta, \gamma) = \left( \frac{1}{2} \mu^2, -\frac{1}{8}(2n - 1)^2, -1 \right)
\]

or
\[
(\alpha, \beta, \gamma) = \left( \frac{1}{8}(2n - 1)^2, -\frac{1}{2}\mu^2, 1 \right)
\]
where $n \in \mathbb{Z}$ and $\mu$ is arbitrary.

- Solutions of $P_V$ satisfying (2) are related to those satisfying (3) by the symmetry
  \[
  S_2 : \quad w_2(z) = 1/w(z), \quad (\alpha_2, \beta_2, \gamma_2, \delta_2) = (-\beta, -\alpha, -\gamma, \delta)
  \]

- Rational solutions of (1) are expressed in terms of the special polynomials $S_n(z; \mu)$ associated with rational solutions of $P_{\text{III}}$. 
Theorem

Suppose that $S_n(\zeta; \mu)$ satisfies the recursion relation

$$
S_{n+1}S_{n-1} = -\zeta \left[ S_n \frac{d^2 S_n}{d\zeta^2} - \left( \frac{dS_n}{d\zeta} \right)^2 \right] - S_n \frac{dS_n}{d\zeta} + (\zeta + \mu)S_n^2
$$

with $S_{-1}(\zeta; \mu) = S_0(\zeta; \mu) = 1$. Then, for $n \geq 1$, the rational solution

$$
u_n(\zeta; \mu) = \frac{S_n(\zeta; \mu)S_{n-2}(\zeta; \mu)}{\mu S_{n-1}(\zeta; \mu + 1)S_{n-1}(\zeta; \mu - 1)}$$

satisfies

$$
\frac{d^2 \nu_n}{d\zeta^2} = \left( \frac{1}{2\nu_n} + \frac{1}{\nu_n - 1} \right) \left( \frac{d\nu_n}{d\zeta} \right)^2 - \frac{1}{\zeta} \frac{d\nu_n}{d\zeta} + \frac{4(\nu_n - 1)^2}{\zeta^2} \left( \alpha_n \nu_n + \beta_n \right) + 2\gamma_n \nu_n
$$

with parameters given by

$$\left( \alpha_n, \beta_n, \gamma_n \right) = \left( \frac{1}{2} \mu^2, -\frac{1}{8}(2n - 1)^2, -1 \right)$$

Remark

- The recurrence relation (1) is the same as for rational solutions of $P_{\text{III}}$. 

Interlacing of Roots?

Do these special polynomials possess an analogous property to well-known the interlacing property for classical polynomials, such as Hermite polynomials $H_n(z)$, Laguerre polynomials $L_n(z)$, Legendre polynomials $P_n(z)$ or Tchebychev Polynomials $T_n(z)$?

- For a set of orthogonal polynomials $\varphi_n(z)$, for $n = 0, 1, 2, \ldots$, if $z_{n,m}$ and $z_{n,m+1}$ are two successive roots of $\varphi_n(z)$, i.e. $\varphi_n(z_{n,m}) = 0$ and $\varphi_n(z_{n,m+1}) = 0$, then $\varphi_{n-1}(\zeta_{n-1}) = 0$ and $\varphi_{n+1}(\zeta_{n+1}) = 0$ for some $\zeta_{n-1}$ and $\zeta_{n+1}$ such that $z_{n,m} < \zeta_{n-1}, \zeta_{n+1} < z_{n,m+1}$.

- The derivatives $\varphi'_n(z)$ and $\varphi'_{n+1}(z)$ also have roots in the interval $(z_{n,m}, z_{n,m+1})$, that is $\varphi'_n(\xi_n) = 0$ and $\varphi'_{n+1}(\xi_{n+1}) = 0$ for some $\xi_n$ and $\xi_{n+1}$ such that $z_{n,m} < \xi_n, \xi_{n+1} < z_{n,m+1}$.
Hermite Polynomials $H_n(z)$

Legendre Polynomials $P_n(z)$

Laguerre Polynomials $L_n(z)$

Tchebychev Polynomials $T_n(z)$
Roots of $Q_{24}$ (red) and $Q_{25}$ (blue)
Roots of $Q_{24}$ (red), $Q_{25}$ (blue) and $Q_{26}$ (green)
Poles of $u_{19}$ (red) and $u_{20}$ (black)
Roots of $H_{6,6}$ (black), $H_{7,6}$ (blue), $H_{6,7}$ (red), $H_{7,7}$ (green) 

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Roots of $Q_{5,5}$ (black), $Q_{6,5}$ (blue), $Q_{5,6}$ (red), $Q_{6,6}$ (green)

P\textsuperscript{IV}

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Conclusions

• Have studied properties of special polynomials associated with rational solutions of P_{II}–P_{V}. The roots of these polynomials have a very symmetric, regular structure.

• The polynomials are defined by second order, bilinear differential-difference equations which are equivalent to the Toda equation. These polynomials also satisfy fourth order bilinear ordinary differential equations and homogeneous difference equations.

• Symmetric structures also arise for the roots of special polynomials associated with rational solutions of the equations in the P_{II} hierarchy (triangles, pentagons, septagons, ... ) and equations in the symmetric P_{IV} hierarchy.

• This seems to be yet another remarkable property of the Painlevé equations, indeed more generally of “integrable” differential equations.
Some Open Problems

• Is there an analytical explanation and interpretation of these computational results?
• Is there an interlacing property for the roots of these special polynomials in the complex plane?
• Is there a generating function $\Psi(z, \lambda)$ for these special polynomials such that
  $$\sum_{n=0}^{\infty} Q_n(z) \lambda^n = \Psi(z, \lambda)?$$
• Are these special polynomials orthogonal either on a (complex) curve $\Gamma$, i.e.
  $$\int_{\Gamma} \omega(\zeta; z) Q_n(z) Q_m(z) \, dz = 0, \quad m \neq n$$
  for some weight function $\omega(\zeta; z)$, or in a domain of the complex plane?
• What are the dynamical systems, with respect to the parameters, for the special polynomials associated with rational solutions of $P_{III}$ and $P_V$?
• What is the structure of the roots of the special polynomials associated with rational solutions of $P_{VI}$ and discrete Painlevé equations?
• Do these special polynomials have applications, e.g. in numerical analysis?